A UNIVERSAL LAW OF GRAVITATIONAL DEFORMATION FOR GENERAL RELATIVITY[a]

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The notion of *universal deformation law* is introduced and its essential elements are drawn. The hypothesis that gravity *acts* as a universal deformation law complements standard General Relativity with concepts of strong heuristic content. A *particular* universal law of gravitational deformation is proposed, which allows to locally separate the gravitational field from space-time frames. Some of its properties are mentioned and the Schwarzschild case is analysed.

I REMEMBER...

Twenty six years ago, in a difficult moment, I adressed to Lluís BEL a letter in which, among other thinks, I told him about my main subject of research, the construction of a *good* theory of accelerated observers, and I asked him to help me in the obtention of a grant. Unfortunately, I was not able to conclude successfully my work but, fortunately, he was able to obtain a grant for me.

I am glad to remember now that letter, in part because the present work answers in some extend my old main subject, but specially because, due to Lluís' help, I could continue working in relativity and having many interesting discussions with him, and I can today be present, among you, in this meeting.

1 INTRODUCTION

This work concerns *standard* General Relativity. It is not a generalization, neither an alternative, nor a correction to Einstein theory. It is simply a *complement* to this theory of gravitation.

To the standard version of the theory, one adds here the hypothesis that *gravitation acts locally as a universal mechanism of deformation.* This hypothesis allows to separate the gravitational field from the local space-time frames.

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It is also proposed here a *particular* mechanism, involving a *universal law of gravitational deformation* in which the gravitational field is described by a second order *antisymmetric* tensor.

To contrast this law, Schwarzschild solution is obtained from it. In spite of its simplicity, this law raises questions of great heuristic interest about matter and gravitational field and *induces* us to ask about the correct answer.

2 THE NOTION OF UNIVERSAL DEFORMATION

Let us imagine a machine, like the one represented in Fig. 1, endowed with a *deformation mechanism* which works as follows: for *every* particular choice of values $\mathbf{F}$ of a set of control parameters, it transforms a flat sheet, of intrinsic flat metric η, in a curved sheet, of intrinsic curved metric g.

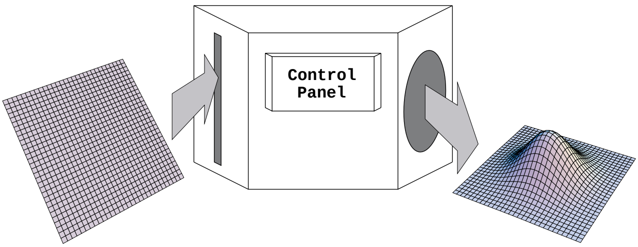

Figure 1: For every set $\mathbf{F}$ of control parameters, this machine transforms a flat sheet of intrinsic flat metric η in a curved sheet of intrinsic curved metric g.

To such a deformation mechanism is associated a *deformation law* $\mathbf{G}$, namely that given by the relation

$$g = \mathbf{G}(\mathbf{F}, \eta) \tag{1}$$

A deformation law is said to be *universal* if it is able to *locally* generate *any* prescribed curved surface, *i.e.*, if, for *any* local pair $\{g, \eta\}$, there always exists a set of values $\mathbf{F}$ of control parameters such that the above relation 1 holds.

Perhaps the best known example of a universal deformation law in two dimensions is the *conformal* one, given by

$$g = f(\mathbf{F})\eta \ , \tag{2}$$

the universality of this law comming from the well known result that any two-dimensional metric is locally flat, $g = \sigma\eta$.

The conformal law is undoubtedly widely known, but many others universal deformation laws exist or may be constructed in two dimensions. For instance, the *parallel deformation law*, of the form

$$g = \eta + f(\mathbf{F}) c \otimes c \ , \tag{3}$$

where c is a η-unit covariantly constant vectorfield and $f(\mathbf{F}) + 1 > 0$; or the *vectorial deformation law*, of the form

$$g = \eta + k u \otimes u \ , \tag{4}$$

where u is a η-unit vectorfield and $k + 1 > 0$.

It is important to remark that, even in this simple two-dimensional case, although the set $\mathbf{F}$ of strict control parameters is one-dimensional, *its tensorial character is not given*: as the above examples show, for conformal and parallel deformation laws it can be a *scalar*, $\mathbf{F} \equiv \{\sigma\}$, whereas for vectorial deformation laws it is a η-unit vectorfield, $\mathbf{F} \equiv \{u \mid \eta(u, u) = 1\}$.

The notion of (local) universal deformation law extends straightforward to n dimensions, the terms of the equation 1 being well defined for any n. Only the form of the universal laws and the set of (strict) control parameters $\mathbf{F}$ change, this last one becoming $\{n(n-1)/2\}$-dimensional.

3 GRAVITATION AS A UNIVERSAL DEFORMATION LAW

It is clear that the notion of universal deformation law, related to equation 1, not only is independent of the dimension of the space, but also of its signature: universal deformation laws are able to act (locally) on Minkowski space, deforming it to a space of non-vanishing curvature.

One can thus consider the following question:

is gravitation a universal mechanism of deformation?

or, in other words, can the gravitational field be characterized by a set $\mathbf{F}$ of parameters such that the space-time curved metric g be locally *nothing but* the result of the action of $\mathbf{F}$ on the Minkowski metric η: $g = \mathbf{G}(\mathbf{F}, \eta)$?

General Relativity is a *causally complete* theory of gravitation, that is to say, a theory with a number of *concepts*, *quantities* and *laws* necessary and sufficient to control their local evolution. To complement this theory with the hypothesis that gravitation is a universal mechanism of deformation, amounts to *add* to it *new* concepts (that of universal mechanism), *new* quantities (the control parameters $\mathbf{F}$) and *new* laws (the associated universal deformation law). Is it worthwhile? As an indirect answer, let me remember two historical situations structurally similar to the present one:

– in Classical Mechanics, the three principles of Newton were the basis of a (causally complete) theory for material points in terms of forces, masses and kinematical quantities. Its complementation with the internally unnecessary concept of *Hamiltonian action* gave rise to the well known different and fruitful vision of the Analytical Mechanics.

– in Classical Electrodynamics, the laws of Coulomb, Biot and Savart, Ampère and Faraday constituted a (causally complete) theory for electromagnetic phenomena in terms of charges, currents and forces. Its complementation with Faradays' internally unnecessary concept of *field* (and the correction of Ampère's law by the displacement current) lead Maxwell to his celebrated equations.

These two historical examples constitute a strong incentive to try to complement General Relativity with a universal law of gravitational deformation and to explore its consequences. A first sketch is presented in what follows.

4 A UNIVERSAL DEFORMATION LAW FOR GRAVITATION

As seen in Section 2, the construction of a deformation law of the form:

$$g = \mathbf{G}(\mathbf{F}, \eta)$$

implies the choice of the number and tensorial character of the parameters of the set $\mathbf{F}$.

The strict number of parameters of a universal law in a n-dimensional space-time being $N = n(n-1)/2$, these N parameters may form a set of N scalars, or of $n-1$ orthonormal vectors (a frame), or a second order tensor submitted to $n(n+1)/2$ constraints, etc.

But here $\mathbf{F}$ has to represent the *gravitational field*, as seen from η, a notion that we want to be directly represented by a *sole*, *covariant* and *a-metric* object. Thus, the elements of $\mathbf{F}$ have to be the components F_{ij} of a two-form F:

$$\mathbf{F} \equiv \{F_{ij} \mid F = F_{ij} dx^i \wedge dx^j\}$$

The deformation law we are asking for is thus of the form:

$$g = \mathbf{G}(F, \eta) \quad . \tag{5}$$

where η is the Minkowski metric, F is the new gravitational field two-form and g is the Einstein metric, *i.e.* the result of the deformation of η by F.

Fortunately, the algebraic properties of two-forms restrict drastically the general form of such a law; in particular, for low dimensions one may show:

Theorem: *For dimensions $n < 6$, the more general* deformation law *induced by a two-form is necessarily of the form:*

$$g = \lambda(F)\eta + \mu(F)F^2 \quad , \tag{6}$$

where λ and μ are scalar functions of F.

The following examples show the "familiar" character of this law:

– In two dimensions two-forms are necessarily proportional to the volume element ϵ: $F = f\epsilon$, f being its sole invariant scalar; consequently, our law 5 becomes in this case

$$g = \Phi(f)\eta \quad , \tag{7}$$

which is nothing else but the well known *conformal* law.

– In three dimensions two-forms are equivalent to vectors: $F = *v$, $*$ being the Hodge dual operator; the law 5 may then be written:

$$g = \lambda(v^2)\eta + \mu(v^2)v \otimes v \quad , \tag{8}$$

a *scalar-vector* deformation law that, twenty-two years ago, I showed to be *universal*.

– In dimension four, for the particular case of *null* two-forms, $F = e \wedge \ell$, with $\ell^2 = \ell.e = 0$, the law reduces to

$$g = k\eta + H(e^2)\ell \otimes \ell \quad , \tag{9}$$

(k = constant), namely a (homothetic) *Kerr-Schild* transformation.

Let us remark that, the funcions $\lambda(F)$ and $\mu(F)$ being scalars, they depend but of the scalars of F. Thus, in four dimensions one has $\lambda = \lambda(\alpha, \beta)$ and $\mu = \mu(\alpha, \beta)$, where α and $i\beta$ are eigenvalues of F.

As said above, the point of view here is that the curved metric g of any *local* portion of the space-time is nothing but the result of the deformation, by the gravitational field F, of the flat metric η. So, an important point is to know the conditions under which the deformation law preserves the signature; one has the result:

Theorem: *The metrics g and η related by*

$$g = \lambda\eta + \mu F^2$$

have same signature if, and only if, λ and μ have the structure:

$$\lambda = \beta^2 p^2 + \alpha^2 q^2 \quad , \quad \mu = p^2 - q^2$$

where p and q, $pq \neq 0$, are arbitrary functions of α and β.

In General Relativity light follows null geodesics of the metric g with g-velocity c. What about its η-velocity c_η? One has:

Theorem: *The velocity of light measured with η, c_η, verifies:*

$$c_\eta \leq c \quad \Longleftrightarrow \quad \mu < 0$$

$$c_\eta \geq c \quad \Longleftrightarrow \quad \mu > 0 \ .$$

5 THE SCHWARZSCHILD SOLUTION

Null gravitational field: the Kerr-Schild form

In a system of *Minkowskian spherical coordinates* $\{t, s, \theta, \varphi\}$, radial null directions ℓ_ε, $\varepsilon = \pm 1$, are given by $\ell_\varepsilon = dt + \varepsilon ds$, and the corresponding null two-forms by:

$$F_\varepsilon = \ell_\varepsilon \wedge p \quad ,$$

where p is the *gravitational polarization vector,* $\eta(p, \ell_\varepsilon) = 0$.

Because the invariants of F vanish, one must have $\lambda(0,0) = 1$ and $\mu(0,0) =$ *constant*, so that the gravitational deformation law in the Schwarzschild case takes the form of the Kerr-Schild transformation:

$$g_{Sch} = \eta + \mu \mid p \mid^2 \ell_\varepsilon \otimes \ell_\varepsilon \quad .$$

Considering it as an equation for p, its solution is given by:

Theorem: *The gravitational polarization vector p of a null gravitational field F_ε generating Schwarzschild metric is of the form $p = ad\theta + bd\varphi$, where a is any function submitted to the* sole *inequality: $a^2 \geq 2\frac{m}{\mu}s$ and b is then given by: $b = \pm\sqrt{2\frac{m}{\mu}s - a^2}\sin\theta$.*

Remark 1: Obviously, neither the polarization p can be regular on the spheres $t = Constant$, $s = constant$, nor the null gravitational field F_ε can have spherical symmetry. Consequently, as a particular case of our deformation law, *Kerr-Schild generation of Schwarschild space-time cannot have physical meaning but for angular sectors.*

Remark 2: Like our eyes with respect to electromagnetic polarization, *the gravitational curved metric g results to be blind with respect to the gravitational polarization*, a property that would be interesting to explore in connection with the measure of gravitational waves.

Regular gravitational field

Denote by $u \equiv dt$ the unit time-like *velocity* of the above Minkowskian spherical coordinate system $\{t, s, \theta, \varphi\}$, and let $e \equiv \alpha(s)ds$ a general *static space-like radial* vectorfield. The general regular static radial spherically symmetric 2-form F is given by:

$$F = u \wedge e = \alpha(s)dt \wedge ds \quad ,$$

and one has:

Theorem: *A regular static radial spherically symmetric gravitational field two-form F generates Schwarzschild exterior solution g_{Sch},*

$$g_{Sch} = \lambda\eta + \mu F^2 \tag{10}$$

if, and only if, $\alpha = \alpha(U)$ with $U \equiv -m/s$. Then, the function $\lambda(\alpha, 0)$ is given by:

$$\lambda = 4U^2(1 + W)^2 \tag{11}$$

where $W(z)$ is Lambert's function and $z \equiv Qexp\{-1/2U\}$.

Lambert's function $W(z)$ is defined by $z = W(z)\exp W(z)$. The conformal factor $\lambda(\alpha, 0)$ in the deformation law 10, given by 11, is shown in Fig. 2.

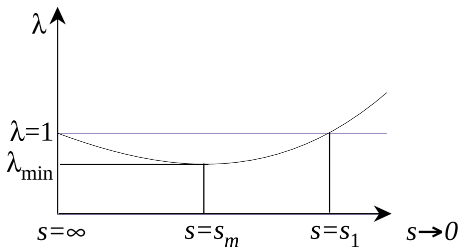

Figure 2: The conformal deformation factor λ is a *dilatation* from $s = 0$ to $s = s_1$ and a *contraction* from $s = s_1$ to $s \to \infty$.

If $\{t, r, \theta, \varphi\}$ are the Schwarzschild spherical coordinates, another interesting result is the following one:

Theorem: *The radial Minkowski coordinate s and the associated radial Schwarzschild coordinate r are related by:*

$$r = 2m[1 + W(Q\exp s/2m] \quad . \tag{12}$$

In the (s, r)-plane, this function has the form shown in Fig. 3. These results allow to conclude with the following remarks:

Remark 3: *The conformal deformation factor λ, which affects Minkowski spheres $t =$ constant, $s =$ constant, is a* dilatation *from the position of the point particle $s = 0$ to a distance $s = s_1$ defined by:*

$$s_1^2 = 4m^2[1 + W(Q \exp s_1/2m] \quad .$$

And it is a contraction *from this distance s_1 to infinity, with a* minimum s_m *between these two last values.*

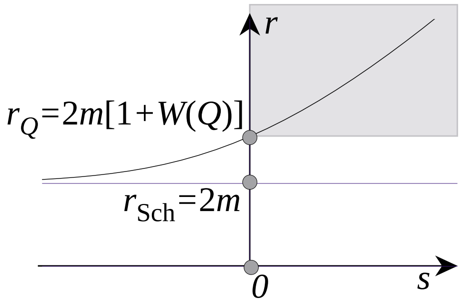

Figure 3: A massive point particle creates only the Schwarzschild space-time region $r > r_Q$, with $r_Q \equiv 2m[1 + W(Q)] > 2m$.

Remark 4: According to our deformation law, *a massive point particle creates the Schwarzschild space-time region corresponding* only *to $r > r_Q$*, where $r_Q \equiv 2m[1+W(Q)]$; in other words, the Schwarzschild "sphere" $r = r_Q$ reduces to the point $s = 0$ where the point mass is located. Thus, for this deformation law, Schwarzschild horizon is a mathematical fiction; its non vanishing *conceptual distance* to the physical notion of *point particle*, $r_Q - r_{Schw} = 2mW(Q)$, depends on the strictly positive value of the *gravitational compressibility* coefficient Q of matter.

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