# About deformation and rigidity in relativity

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**Abstract.** The notion of deformation involves that of rigidity. In relativity, starting from Born's early definition of rigidity, some other ones have been proposed, offering more or less interesting aspects but also accompanied of undesired or even pathological properties. In order to clarify the origin of these difficulties presented by the notion of rigidity in relativity, we analyze with some detail significant aspects of the unambiguous classical, Newtonian, notion. In particular, the relative character of its kinetic definition is pointed out, allowing to predict and to understand the limitations imposed by Herglotz-Noether theorem. Also, its equivalent dynamic definition is obtained and, in contrast, its absolute character is shown. But in spite of this absolute character, the dynamic definition is shown to be not extensible to relativity. The metric deformation of Minkowski space by the presence of a gravitational field is interpreted as a universal deformation, and it is shown that, under natural conditions, only a simple deformation law is possible, relating locally, but in an one-to-one way, gravitational fields and gauge classes of two-forms. We argue that fields of unit vectors associated to the internal gauge class of two-forms of every space-time (and, in particular, of Minkowski space-time) are the relativistic analogues of the classical accelerated observers, i.e. of the classical rigid motions. Some other consequences of the universal law of gravitational deformation are commented.

## 1. Introduction

There exists no real rigid bodies. Even in Newtonian mechanics, under the effect of accelerations, be they time-varying or constant, material bodies deform, regardless of their chemical constitution. But rigidity is a helpful notion in many different situations, and has been fruitfully applied in physics with many different senses.

*Rigidity as a concept*

Depending on the particular context, rigidity has been seen as:

- an *approximate real* concept. Like that of a set square for technical drawing with respect to an ideal mathematical triangle, rigidity may be considered the property of approximate physical materializations (*rigid bodies*) of ideal motions of point-like particles whose mutual distances remain ideally constant.

- a *paradigmatic exact ideal* concept. Like that of inertial observers, free particles or monochromatic plane waves, rigidity is sometimes considered as an unreal but paradigmatic concept, whose importance rather lies in its role as an ideal reference or basis for the construction and control of more complicated but more realistic notions (elasticity, plasticity and so on).

- a *virtual* concept. Like that of the most part of Cartesian coordinate system. Be they drawn on a sheet of paper or constructed on a building site, only very few coordinate lines are really carry out to define it (frequently only their coordinate axis) and to use it (only the coordinate lines converging at the points of interest); in this sense, Cartesian coordinate systems are essentially virtual and so is also the rigidity associated to their ingredients.

- a *material* concept. Like that of the bimetallic gridiron rod of a pendulum clock to compensate temperature expansion, one could think of rods with special structures allowing them to remain rigid under some range of applicable forces. For example, telescopic rods endowed with accelerometers and gyrometers to control their length and a cable system to control their straighteness (Figure 1).

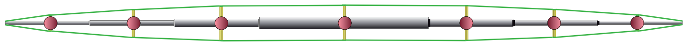

**Figure 1.** A telescopic rod endowed with accelerometers and gyrometers to control its length and a cable system to control its straighteness.

Free structures constructed with such rods cannot detect deformations, even if they are 'geometrically rigid', like triangles. But overdetermined structures can do it. In the case of Figure 2 (b), this can be made measuring the strain or the separation at the contact ends of the large rods, the short ones being kept in permanent contact.

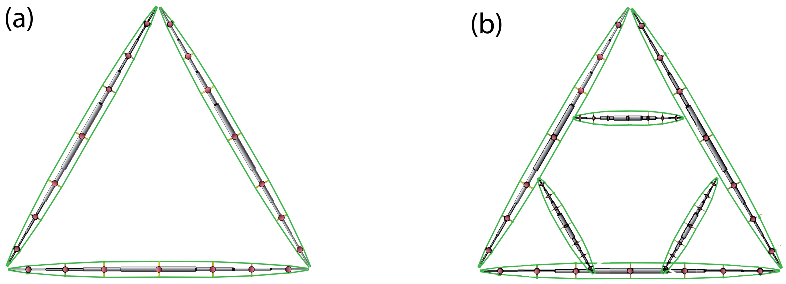


**Figure 2.** (a) free structures cannot detect deformations. (b) over-determined structures can do it.

Similar questions and corresponding devices for accelerated time-keeping clocks can be asked. In this case, the adequate term for the clocks is *uniform* in place of *rigid*, and two clocks will be uniform if, submitted to different arbitrarily accelerated motions around an observer, they remain synchronized.

For Newtonian physics such structures exist which remain rigid and uniform respectively. But for relativistic physics, there is at present *no result* allowing to answer whether or not such rigid and uniform structures can be constructed. This work try to introduce the ingredients for a precise answer to this question.

*Rigidity as the space of states of motion of a laboratory*
Perhaps the main application of the notion of rigidity in Newtonian physics is the one that, as a virtual concept, allows to define the *space of states of motion* of a laboratory.

A laboratory (represented in Figure 3 by a spacecraft constructed with rigid devices) can be arbitrarily accelerated. To honor the *objective* character of physical phenomena, we must be able to construct and to describe real or thought experiments in any of its arbitrarily accelerated states and to find the invariant properties of the phenomena involved.

In Newtonian physics, the space of all the possible states of motion of the laboratory is nothing but the space of rigid motions. Unfortunately, in relativity the notion of *laboratory* has not been still constructed, principally because, following the classical way, one has tried to identify it with a virtual rigid motion, and they are not known rigid motions in relativity which could be arbitrarily accelerated.

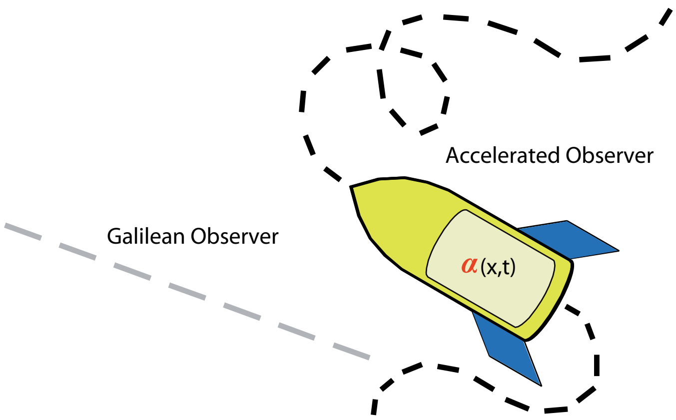


**Figure 3.** A laboratory may be represented by a spacecraft constructed with rigid devices and arbitrarily propelled.

The main objective of the present work is to introduce in relativity the ingredients necessary for the construction of a notion of laboratory.

*rigidity as an unnecessary concept*
Well, but why this objective? Why to try to introduce a notion that many people feels problematic if not contradictory, and any way seems to believe unnecessary?

General Relativity is a *complete* theory in the sense that :
- its geometric frame (differentiable manifold with metric) ,
- its specific variables (metric components) and
- its structure system (Einstein equations)

are necessary and sufficient to associate a unique solution to standard initial data; in other words: the theory has *no need* of new elements or concepts for its development. In this sense, it is true in particular that, as well as the other concepts which are attached to it, *rigidity is an unnecessary concept for general relativity.*

Thus, why to try to introduce it in relativity? They are, in fact, many inferential reasons to justify *a priori* such an attempt, but they are neither easy not short to explain. For this reason, we will limit ourselves here simply to argue by analogy, remembering some well known historical cases where the introduction of unnecessary concepts in a complete theory has given a fruitful and completely different vision of the theory, enriching it with new theoretical and experimental conceptions and enlarging its domains of applications.

- This is the case, for example, for the concept of *Hamiltonian action.* The three Newton principles make Classical Mechanics a complete theory for material points in terms of forces, masses and kinetic quantities. The supplementary concept of Hamiltonian action, structurally unnecessary, has drastically changed the conception of the theory, as every one knows, endowing it with a powerful structure of unforeseen applications in modern physics.
- Another unnecessary but extremely fertile concept was that of *physical field.* Coulomb, Biot et Savart, Ampère and Faraday laws constitute a complete theory for electromagnetic phenomena in terms of forces, charges and currents. The supplementary concept of field (apart from the displacement current) led Maxwell to his celebrated equations, whose structure has been at the basis of relativistic and quantum physics.

On account of these and other examples and of the physical interest that could have the extension to relativity of the classical notion of accelerated laboratories as specific generalizations of inertial ones, we will consider here as worthwhile the efforts to try to introduce in relativity something analogous to classical rigidity.

## 2. KINETIC NOTION OF RIGIDITY

The first notion of *rigid motion* is a kinetic one: a motion of a system is rigid if the distance between any two points of the system remains constant, irrespective of the cause that produce the motion. The purpose of this section is to compare the kinetic definitions of Galilean and relativistic rigidity. This will allow us to understand that, in spite of its appearance, the relativistic one cannot be considered as an extension of the Galilean one.

*Galilean rigidity*

In order to make explicit the structure of Galilean rigidity, let us briefly remember the structure of the corresponding space-time.

Galileo space-time is constituted by the four-dimensional extension $\mathtt{R}^4$ endowed with two degenerate metrics, a *time-like* metric $T$ and a *space-like* metric $\gamma^*$. The time-like metric $T$ is a rank 1, signature 1, covariant metric so that it can be written as the tensor square of a one-form $C$, the *time-like current*, $T = C \otimes C$. The space-like metric $\gamma^*$ is a rank 3, signature 3, contravariant metric orthogonal to $T$: $i(C)\gamma^* = 0$, where $i(.)$ stands for the interior product.

The existence of an *absolute time* implies the exact character of $C$, $C = dt$, where $t$ is any absolute time scale[1]. The hypersurfaces $t = Constant$ constitute the instantaneous spaces or spaces at the instant $t$.

A vector $v^*$ is said *time-like* if $i(C)v^* \neq 0$; otherwise it is called *space-like*. A time-like vector $u^*$ is a *unit* vector if $i(C)u^* = 1$. Every time-like unit vector $u^*$ has associated a unique rank 3, covariant, space-like metric $\gamma = \gamma(\gamma^*, u^*)$, the one that verifies

$$\gamma \times \gamma^* = I - C \otimes u^* \,. \tag{1}$$

where $\times$ stands for the cross product (contraction of the adjacent spaces of the tensor product). Multiplying (1) first on the left by $u^*$ then on the right by $\gamma$, one obtains $i(u^*)\gamma = \lambda C$ but because $C$ is the null direction of $\gamma^*$ and from (1) $\gamma$ is also of rank 3 in the dual space, $\lambda$ must vanish and one has $i(u^*)\gamma = 0$ : the unit velocity $u^*$ is orthogonal to the covariant metric tensor $\gamma$ associated to it, in a similar way as the time-like current $C$ is orthogonal to the contravariant metric tensor $\gamma^*$.

Galileo space-time stipulates that the regular metric induced by $\gamma$ on every (three-dimensional, instantaneous) space $t = Constant$ is flat.

An *observer* is a system of points whose space-time history is defined by a congruence of time-like curves. Conversely, to every time-like unit vector field $u^*$ one can associate one observer: the system of points whose space-time history is defined by the integral curves of $u^*$.

The privileged synchronization by instantaneous spaces imposed by $C$ allows the observer $O$ associated to any unit vector field $u^*$ to use the standard *3-dimensional formalism.* In this formalism, for a particle of unit velocity $v^*$, its three-dimensional *relative velocity* with respect to $O$ is the space-like vector $\nu^*$, $i(C)\nu^* = 0$, obtained as the quotient of the orthogonal and the tangent projections[2]

$$\nu^* = \frac{\perp(u^*)v^*}{i(C)v^*} = \frac{(I - u^* \otimes C)v^*}{i(C)v^*} = v^* - u^*.$$

With respect to $O$, its unit velocity $v^*$ admits thus the decomposition $v^* = (1, \nu*)$. Note that meanwhile $v^*$ as the unit velocity of a particle, is a vector under the whole group $\mathtt{G}$ of

[1] A time scale is a rhythm generated by a unit interval together with a choice of origin.

[2] Observe that the time-like current $C$, as representation of an absolute clock for any time like-direction, plays here the same role that the covariant 1-forms associated by regular Lorentzian metrics to time-like unit vector $u^*$.

diffeomorphisms of Galileo space-time, $\nu^*$, as the relative velocity of the particle, is a vector only under the group $\mathtt{O}$ that leaves invariant the observer $O$ associated to $u^*$.

Now, in Galileo space-time, it is well known and easy to see that a system of real or virtual particles of relative velocity $\nu^*$ with respect to an observer $O$ of space-like covariant metric $\gamma = \gamma(\gamma^*, u^*)$ follows a rigid motion if $\nu^*$ and $\gamma$ verify:

$$L(\nu^*)\gamma = 0. \tag{2}$$

where $L(\nu^*)$ stands for the Lie derivation with respect to the vector field $\nu *$.

Observe that (2) is an *intrinsically relative* characterization of a rigid motion in the sense that it connects the absolute kinematic ingredient of the rigid system of particles, i.e. its unit velocity $v^*$, to another, not physically connected element, the arbitrarily chosen exterior observer $O$, related to a unit vector field $u^*$ (necessarily different from that $v^*$, of the rigid system, otherwise the relative velocity $\nu^*$ would vanish and the equation (2) would be identically satisfied). But this characterization has the advantage of showing the two roles that the time-like current $C$ plays in the elaboration of a rigid motion:

(a) it gives a unique, privileged, synchronization $t = Constant$, the instantaneous spaces, valid for all the observers,

(b) it allows, *on these integral three-dimensional spaces*, the construction of a space-like flat metric $\gamma$.

In other words, the first notion of rigid motion may be now specified more explicitly: a motion of a system is rigid if, irrespective of its kinematics, the distance between any two points of the system, *measured on the privileged instantaneous spaces with a specific flat metric,* remain constant along its history.

From the above construction, it is then clear that, besides the intrinsically relative characterization (2), one has also the equivalent *absolute characterization* of a rigid motion:

$$L(v^*)\gamma = 0. \tag{3}$$

where now $v^*$ is the (four-dimensional) unit velocity field of the system, in contrast with the (three-dimensional) relative velocity field $\nu^*$ in (2). Note also that in (2) the metric $\gamma$ is the one associated to the observer $u^*$, $\gamma(\gamma^*, u^*)$, meanwhile in (3) it denotes the one associated to the system $v^*$, $\gamma(\gamma^*, v^*)$.

Because of the relation $i(u*)\gamma = 0$, for any function $\mu \neq 0$, (3) gives also $L(\mu v^*)\gamma = 0$. What about the variance of the contravariant metric $\gamma^*$?

From (1) and (3), and taking into account that $i(v^*)C = 1$ and that $i(v^*)\gamma = 0$, one has

$$\gamma \times L(\mu v^*)\gamma^* = -(d\mu - \dot{\mu}C) \otimes v^*$$

where $\dot{\mu}$ stands for $L(v^*)\mu$. Then, multiplying on the left by $\gamma^*$, one obtains:

$$L(\mu v^*)\gamma^* = -v^* \otimes \gamma^*(d\mu) - \gamma^*(d\mu) \otimes v^*. \tag{4}$$

We see that, in contrast with the invariance of the covariant metric $\gamma$ for any $\mu \neq 0$, only for funtions $\mu$ that leave invariant the foliation $t = Constant$, i.e. such that $d\mu \wedge C = 0$, the contravariant metric $\gamma^*$ remains invariant:

$$L(\mu v^*)\gamma^* = 0 \quad \Leftrightarrow \quad \mu = \mu(t). \tag{5}$$

*Relativistic rigidity*

In Minkowski space-time $(\mathtt{R}^4, \eta)$, for every system of unit velocity $v^*$, $\eta(v^*, v^*) = 1$, the two degenerate metrics $T$ and $\gamma^*$ are obtained from the rank 4, Lorentzian metric $\eta$ by:

$$T = v \otimes v \quad , \quad \gamma^* = \eta^* - v^* \otimes v^* \quad , \tag{6}$$

where the time-like current $v$ is given by $v = \eta(v^*)$ and $\eta^*$ denotes the contravariant or inverse metric of $\eta$, $\eta \times \eta^* = I$. The corresponding covariant space-like metric $\gamma = \eta - v \otimes v$, is related to $\gamma^*$ like in the Galilean case (1):

$$\gamma \times \gamma^* = I - v \otimes v^* \, . \tag{7}$$

Of course, in addition to $i(v^*)v = 1$, one has also $i(v^*)\gamma = i(v)\gamma^* = 0$.

In this relativistic context, the standard notion of rigidity is due to Born[3] [1]. It is an *absolute* characterization, and states that the motion of a system is rigid if the above space-like metric tensor remains constant along the history of the system, that is to say if:

$$L(v^*)\gamma = 0. \tag{8}$$

By the same reason that in the Galilean case, for any vector field $\xi^* \equiv \mu v^*$, $\mu \neq 0$, (8) gives also $L(\xi^*)\gamma = 0$. And, as in the Galilean case, in general one has $L(\xi^*)\gamma^* \neq 0$. It is not difficult to show that the contravariant version of (8) may be written as:

$$L(\xi^*)\gamma^* = 2L(\xi^*)\eta^* - v^* \otimes i(v)L(\xi^*)\eta^* - i(v)L(\xi^*)\eta^* \otimes v^*) \quad ,$$

so that it can be easily shown that

$$L(\xi^*)\gamma^* = 0 \quad \Leftrightarrow \quad L(\xi^*)\eta = \{i^2(v^*)L(\xi^*)\eta\}\, v \otimes v \quad . \tag{9}$$

*Comparative: Born and classical rigidity*

Observe that (8) and (3) look identical. This is the reason why many people think that Born notion is correct. But this resemblance is a trap. The advantage of the Galilean relative characterization is that it reveals that the solutions of the absolute equation (3) correspond

(a) to a hypersurface-orthogonal current $C$ for the unit velocity $v^*$ and

(b) to a rank 3 metric tensor $\gamma$ that induces a flat metric on the foliation defined by $C$.

Under such requirements, as it is well known, (3) admits the whole set of Galilean rigid motions. And it is under these assumptions, a flat foliation and a hypersurface-orthogonal current $v$, that equations (8) become identical to equations (3). But then, because the additional constraint $v^* = \eta^*(v)$, equations (8) admit a solution but a unique solution $v^* = \eta^*(v)$ there where equations (3) admits its whole set. We see that in relativity the different irrotational solutions are obtained taking different flat foliations. Thus, specifically, the irrotational solutions $\xi^* \equiv \mu v^*$ to the Born rigid motions (8) are the vector fields associated by the metric $\eta^*$ to the one-forms $\xi$ obtained as gradients of any one parameter family of space-like hyperplanes. Such families may be constructed starting from an arbitrary time-like curve and drawing the

[3] The bibliography on the relativistic notion of rigidity is very extensive in number and large in the variety of the different properties investigated. Even if the most part of the papers concern exact or approximate properties of Born definition in general or in particular space-times, some of them explore the very notion of rigidity itself, and propose different versions to the Born's one. Among them, but not exclusively, one finds [3], [2] [4], [5], [6]. We think, nevertheless, that the present analysis has no direct relation with the most part of this bibliography, apart from the little number of papers indicated in the section References and from the feeling of dissatisfaction on the adequacy of the concept that, globally, the whole bibliography seems to produce.

orthogonal hyperplanes to it. This result constitute one of the two parts of the well known Herglotz-Noether theorem, but with a very simple proof.

Roughly speaking these rigid motions are a half of the Galilean rigid motions. What about the rotational Born rigid motions? For them we cannot now be inspired by the underlying structure associated to the relative Galilean equation (2). But we can consider equations (9).

The Galilean analogues (5) of equations (9) show that Galilean rigid motions are such that the contravariant metric $\gamma^*$ is invariant for all the parameterizations of the trajectories by time scales respecting the absolute synchronization. It is the second part of the Herglotz-Noether theorem, that says that rotational Born motions have necessarily Killing trajectories, which guaranties the same result of invariance of the metric $\gamma^*$.

As a conclusion of this walk on the subject, we believe that in spite of some analogies, from the point of view of their structure, Born and Galilean kinematic notions of rigidity cannot be considered as homologues.

The next step is: what about a dynamic notion of rigidity?

## 3. DYNAMIC NOTION OF RIGIDITY

*Newtonian acceleration field*

*Newtonian space-time* is Galileo space-time endowed with a inertial flat connexion allowing to separate the whole set of rigid observers in two classes:

- inertial observers and
- accelerated observers .

Because inertial observers are well known both in classical physics and in relativity, we will focus here on the accelerated ones.

Let us begin with an inertial observer $I$, and remember that the *position vector* $r(t)$ of a point $P$ is a vector for the group $\mathtt{I}_o$ of transformations of $I$ that leave invariant the origin $O$, that the *relative velocity* $\nu(t) \equiv \dot{r}(t)$ of P is a vector for the whole group $\mathtt{I}$ of transformations of $I$ and that the *relative acceleration* $a(t) \equiv \dot{\nu}(t)$ of P is vector for the whole Galileo group of transformations between inertial observers. This means that relative acceleration is also *absolute,* i.e. an object invariant by the whole set of inertial observers: to know it, different inertial observers have nothing but to express the *same* vector in their different coordinates, what is not the case neiter for the position vector, nor for the velocity. Dynamics being invariant by the Galileo group, a strict dynamic characterization of a system must involve accelerations and its variations with exclusion of positions and velocities.

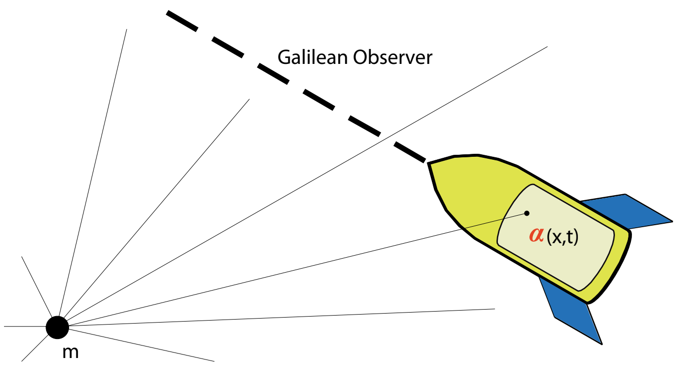


**Figure 4.** How a Galilean observer characterizes dynamically a gravitational field?

An example of such a characterization is given by the field equations of Newtonian gravity. The dynamic characterization of the gravitational acceleration $a$ by a Galilean observer at the exterior of the sources, because of the Laplace equation $\Delta V = 0$ for the potential, $a = -dV$, is obviously given by:

$$da = 0 \quad , \quad \delta a = 0 \quad . \tag{10}$$

What is the dynamic characterization of an inertial field? i.e. what are the corresponding analogues to equations (10) for a inertial acceleration field in Newtonian physics?

It is startling that such a simple question seems not to have aroused any interest up to recently, when it has been answered by Ferrando and myself [7]. To obtain and understand the answer, we need two intermediate results. The first one concerns the necessary and sufficient condition for some tensors to admit a special square root in three-dimensional space:

**Lemma 1.-** *A second order symmetric tensor $L$ admits an antisymmetric square root if, and only if, it verifies:*

$$L^2 - \frac{1}{2}(\mathtt{tr}L)L = 0 \quad .$$

The second one gives the algorithm to obtain it:

**Lemma 2.-** *For such a tensor $L$, its antisymmetric square root is given by*

$$\sqrt{L} = \frac{1}{\sqrt{i^2(x)\overline{L}}} * i(x)\overline{L}$$

*where*

$$\overline{L} \equiv L - \frac{1}{2}(\mathit{tr}L)g \quad ,$$

*the Hodge dual is denoted by $*$ and $x$, submitted to $i^2(x)\overline{L} \neq 0$, is an otherwise arbitrary vector.*

We have now the ingredients to present the dynamic characterization of Newtonian inertial fields. The result is:

**Theorem 1.-** *An acceleration field $a(x,t)$ is an inertial acceleration field corresponding to an accelerated observer if, and only if, it verifies*

$$\nabla L(a)\gamma = 0 \quad , \quad da = \sqrt{2\overset{\bullet}{L(a)\gamma}} \tag{11}$$

*where $\nabla$ denotes the covariant derivative with respect to the metric $\gamma$, $L(a)\gamma$ is the Lie derivative of $\gamma$ with respect to the acceleration $a$ and $\bullet$ stands for the time derivative.*

An inertial field is the field of accelerations that measure an accelerated observer in its rigid laboratory. So its characterizes rigid motions and the above theorem may be alternatively formulated as:

**Corollary 1.-** *A rigid motion is a motion whose acceleration field $a(x,t)$ verifies the system* (11).

It is also interesting to emphasize the common space of solutions that the kinetic system (2) and the dynamic system (11) have:

**Corollary 2.-** *The following differential systems are strictly equivalent:*

$$L(\nu^*)\gamma = 0 \quad \Leftrightarrow \quad \left\{ \nabla L(a)\gamma = 0 \quad , \quad da = \sqrt{2\overset{\bullet}{L(a)\gamma}} \right\} \quad .$$

Now that we have a dynamic characterization of Newtonian rigidity, the question is: Is it worthwhile to try to found a dynamic relativistic generalization of the Newtonian system (11)?

*Characteristics of a relativistic motion*
Like in Newtonian physics, in relativity the position vector $r(t)$ is a vector for the restricted group $\mathtt{I}_o$ of an inertial observer $I$ and the relative velocity $\nu(t)$ is a vector for the whole internal group $\mathtt{I}$ of $I$. But in relativity the relative acceleration $a(t)$, as well as its higher derivatives, remains in $\mathtt{I}$, it does not jump to the corresponding bigger group of transformations between inertial observers, here the Poincaré group $\mathtt{P}$.

Consequently, a generalization to relativity of the Galileo invariant dynamic characterization (11) of accelerated observers should involve, not the relative acceleration,

$$a \equiv \frac{d^2 r(t)}{dt^2} = \frac{d\nu(t)}{dt},$$

but the Poincaré invariant *proper acceleration* a,[4]

$$\mathrm{a} \equiv \frac{d^2\rho(\tau)}{d\tau^2} = \frac{dv(\tau)}{d\tau} = i(v)\nabla v,$$

where $\rho$ and $v$ are respectively the four-dimensional position vector and unit velocity, and $\tau$ is the proper time, $\eta(v,v) = 1$, $\nabla$ being the covariant derivative with respect to the Minkowski metric $\eta$.

Well, is it then *pertinent* to try to find in relativity a system $\mathcal{R}(\mathrm{a})$ playing a similar role to that played in Newtonian theory by the system $\mathcal{N}(a)$ given by (11)?

Certainly not. The system $\mathcal{N}(a)$, is such that, for any of its solutions $a(r,t)$, the second order differential equations

$$\frac{d^2 r(t)}{dt^2} = a(r(t), t)$$

admit a correctly posed Cauchy problem in the Hadamard sense, with a clear physical meaning for all the Cauchy data, as it is well known. In strong contrast, and it is very less known, whatever be the system $\mathcal{R}(\mathrm{a})$, the differential equations associated to their non vanishing solutions $\mathrm{a}(\rho)$ with $\rho \equiv \{r, t\}$,

$$\frac{d^2\rho(\tau)}{d\tau^2} = \mathrm{a}(\rho) \tag{12}$$

admit generically *no solutions*. Only in some cases, *obeying particular constraints*, these differential equations can admit solutions, but then their number is drastically restricted.

The obstruction to the admission of solutions comes from the proper time $\tau$ by the relation

$$\eta(u, u) = 1 \tag{13}$$

which appears as a *fifth* scalar constraint to the four equations (12) in the four variables $\rho$.

Thus, for example, in the simplest case of two dimensions, it is not difficult to prove the following:

**Theorem 2.-** *In a two-dimensional Lorentzian space-time, a space-like vector field* $\mathrm{a}(\rho)$ *is the* proper acceleration *of some system of trajectories if, and only if, it verifies:*

$$i^2(*\mathrm{a})\left\{L(\mathrm{a})\eta - 2\,|\mathrm{a}|^2\,\eta\right\} = 0 \quad . \tag{14}$$

*where* $*$ *stands for the dual Hodge operator and* $|a|^2 \equiv |\eta(a,a)|$.

[4] In relativity, because of the isomorphism between vector fields and 1-forms, we shall denote both with the same symbol.

In particular, a direct but interesting consequence is:

**Corollary 3.-** *In a two-dimensional Lorentzian space-time, any space-like conformal vector field* a *of the form*

$$L(\mathrm{a})\eta = 2\,|\mathrm{a}|^2\,\eta$$

*is the proper acceleration of some system of trajectories.*

And, in any case the first integral is easily given algebraically:

**Theorem 3.-** *In a two-dimensional Lorentzian space-time, the unit velocity $v$, $i(v)\nabla v = \mathrm{a}$, of a proper acceleration field* a, i.e. *such that*

$$i^2(*\mathrm{a})\left\{L(\mathrm{a})\eta - 2\,|\mathrm{a}|^2\,\eta\right\} = 0 \quad ,$$

*is given by:*

$$v = \epsilon \frac{1}{|\mathrm{a}|} * \mathrm{a} \quad .$$

*where $\epsilon$ is the sign for $v$ to be future oriented.*

This two-dimensional result has the advantage of showing the structure of the general problem and how drastically, in the best of the cases, the number of first integrals is restricted.

In the four-dimensional Minkowski space-time, let us associate, to any arbitrary vector field a, the vector field differential concomitant $\alpha(\mathrm{a})$ and the tensor field differential concomitant $A(\mathrm{a})$ given by :

$$\alpha(\mathrm{a}) \equiv i(\mathrm{a})\nabla \mathrm{a} + \frac{3}{2}\nabla \mathrm{a}^2 \quad ,$$

$$A(\mathrm{a}) \equiv 2\left\{\nabla\nabla \mathrm{a}^2 + i(\mathrm{a})\nabla\nabla \mathrm{a}\right\} + \nabla i(\mathrm{a})\nabla \mathrm{a} - \nabla \mathrm{a} \times^t \nabla \mathrm{a} \quad ,$$

where now $\nabla$ denotes the covariant derivative associated to the connexion of the space-time and ${}^t(\nabla \mathrm{a}) \equiv {}^t\nabla \mathrm{a}$ denotes the transposed tensor. Consider the following algebraic system in the vector unknown $v$ :

$$\left.\begin{array}{l} i(v)v = 1 \\ i(v)\mathrm{a} = 0 \\ i^2(v)\nabla \mathrm{a} = -\mathrm{a}^2 \\ i^3(v)\nabla\nabla \mathrm{a} + i(v)\alpha(\mathrm{a}) = 0 \\ i^4(v)\nabla\nabla\nabla \mathrm{a} + i^2(v)A(\mathrm{a}) = -i(\mathrm{a})\alpha(\mathrm{a}) \end{array}\right\} \qquad (15)$$

Then one can show :

**Theorem 4.-** *In Minkowski space-time, an arbitrary vector field* a *is a* proper acceleration field *if, and only if, it verifies the compatibility conditions in* a *for the system* (15) *to admit a solution in $v$. Then, the system of trajectories is the system of integral curves of the algebraic solutions $v$ to* (15).

The problems of finding, in terms of the sole vector field a and its differentials, the compatibility conditions for the system (15) to admit a solution in $v$, as well as that of finding the explicit and covariant expression of $v$, are still open.

Anyway, what the above results show is that there is no meaning to try to extend to relativity the classical dynamical notion of rigidity under the form of a system of equations for an acceleration field *on* the sole space-time. Under a less explicit form, this result is, in fact, well known: as it is manifest in electromagnetism, an acceleration field must at least be a field of the vector bundle of time-like fields of directions on the space-time, i.e. it must explicitly depend on the space-time events *and* on the unit velocity of the particles in question: $\mathrm{a} = \mathrm{a}(\rho, v)$. The electromagnetic case, $\mathrm{a}(v) = F\,(v)$, is the simplest particular case.

It remains that every precise notion of deformation involves a notion of rigidity. It is this clue that we shall consider in what follows .

## 4. RIGIDITY AS ABSENCE OF DEFORMATION

*A universal law of gravitational deformation*

(a) Notion of gravitational deformation. In 1913, for his four-dimensional calculation of the deviation of light rays in a gravitational field, Einstein started from Minkowski metric

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2dt^2 \quad ,$$

and *deformed* it by a scalar gravitational potential $\varphi$ in the form:

$$ds^2 = dx^2 + dy^2 + dz^2 - c^2(1 + \varphi/c^2)dt^2 \quad ,$$

which is of the form

$$g = \eta + h \tag{16}$$

This expression says that the presence of a gravitational potential $\varphi$ *deforms* the flat metric $\eta$ by an additive term $h = \varphi dt^2$, transforming it in a generically non flat metric $g$. In this conception, (16) is the *deformation law* followed by this process[5].

But in his General Relativity of 1916, his field equations $\mathrm{Eins}(g) = \chi T$, relate intrinsically the non flat metric $g$ to the energy content: they no longer refer to any non gravitationally deformed metric $\eta$. Thus, Einstein equations give directly the *gravitationally deformed* metric, but give not the *deformation law* $\mathcal{D} : \eta \to g$ .

Can it be given a precise notion of metric deformation? The answer is yes. To begin with, let us first consider a two dimensional situation, like the one shown in Figure 4. Suppose a machine

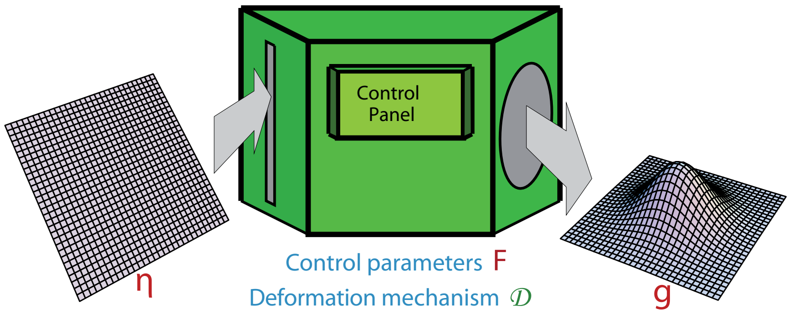


**Figure 5.** A machine endowed with a deformation mechanism $\mathcal{D}$ deforms flat sheets according to the choice of a set F of parameters.

endowed with a deformation mechanism $\mathcal{D}$ controlled by a set of parameters F. According to the choice of these parameters, the machine transforms a flat sheet, of intrinsic flat metric $\eta$, into a deformed sheet, of intrinsic curved metric $g$. The device may be thus represented by a function of the form:

$$g = \mathcal{D}(\mathrm{F}, \eta) \quad . \tag{17}$$

[5] This process may be imagined as a global 'ontologic' one, which happens in the space of all possible universes, or as a local 'epistemic' one, which is physically experimented by an observer in his local laboratory who would be traveling from an asymptotic region where $g \approx \eta$ under the experimental uncertainties, toward the masses that create the field, where $g$ appears no longer flat.

We shall call such a function a *deformation law.* A deformation law is said *universal* if it can locally generate any surface, i.e. if for any choice of a local pair $(g, \eta)$ it always exist parameters F such that $g = \mathcal{D}(\mathrm{F}, \eta)$ takes place.

A simple example of a universal deformation law in two dimensions is the conformal law $g = f\eta$, where the set F of parameters is reduced to the scalar $f$.

The local notion of universal deformation law (17) is clearly independent
- of the dimension of the space and
- of the signature of the metric,

so that, in principle, it can be applied in particular to the space-times $(V_4, g)$ of General Relativity with respect to Minkowski space-time $(\mathtt{R}^4, \eta)$. The question is:

*does gravitation act as a physical universal deformation law?*

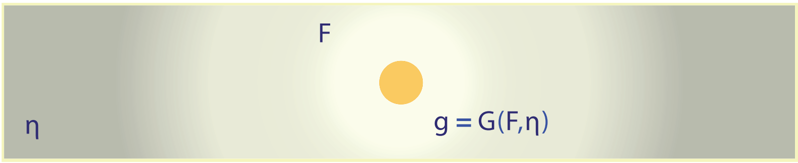


**Figure 6.** Is the gravitational field characterized by an object F such that the space-time metric $g$ be the result of the action of F on Minkowski metric $\eta$ ?

(b) Our universal law of gravitational deformation. The construction of a deformation law $\mathcal{D}$ of the form (17) implies the specification of the number of independent parameters $N$ of F and of its tensor character.

The number $N$ of independent parameters to generate locally any metric in a $n$-dimensional space-time is necessarily $N = n(n-1)/2$ by the well known Riemann's theorem on the degrees of freedom of a metric. And the tensor character of F is fixed by this value of $N$ and our two requirements that F must constitute a *single geometric object* and that it must be a *non metric object,* i.e. the restriction to $N$ independent components for F must follow from constraints involving the sole F, and not the metric. This hypothesis seems nice because gravitational fields exist and their effects are perceptible irrespective of our metrical conventions and the uncertainties of our metric constructions.

It is then easy to see that, under the two above requirements, the set of parameters F must constitute a two-form $F$,

$$\mathrm{F} = \left\{ F_{ij} \middle| F = F_{ij} dx^i \wedge dx^j \right\}.$$

We are thus looking for our deformation law $\mathcal{D}$ under the form :

$$g = \mathcal{D}(\mathcal{F}, \eta) \quad , \tag{18}$$

where now we know that $F$ is a 2-form. The first simple but important result is the following one:

**Theorem 5.-** *For dimension $n < 5$, the deformation law $g = \mathcal{D}(\mathcal{F}, \eta)$ is necessarily of the form*

$$g = \lambda \eta + \epsilon F^2 \quad , \tag{19}$$

*where $\lambda$ is a function of $F$, $\lambda = \lambda(F)$, $\epsilon$ is a prefixed sign and the square of $F$ denotes the cross product induced by $\eta$, $F^2 \equiv F \times \eta^* \times F$.*

In fact, for lower dimensions, this law is not unknown, as show the following examples:

- in dimension two, two-forms are proportional to the volume element, $F = f\mathrm{v}$, and consequently the law becomes:
$$g = \phi(f)\,\eta\,,$$
i.e. the well known conformal deformation,
- in dimension three, two-forms are equivalent to vectors, $F = *v$, and the law becomes:
$$g = \lambda(v)\,\eta. + \epsilon\, v \otimes v \quad ,$$
a scalar-vector law that I proved 25 years ago for a particular gauge and has been proved in general gauge in [8],
- in dimension four, for null two-forms , $F = e \wedge \ell$, $e.\ell = \ell.\ell = 0$, one has
$$g = K\eta + H(e)\,\ell \otimes \ell \quad ,$$
where $H(e)$ is a scalar function of the polarization vector $e$ and $K = Constant$, i.e. a Kerr-Schild transformation.

Theorem 5 says us that a law of the form (18) is necessarily of the form (19) for dimensions two, three and four but not further on. On the other hand, we already know that the law (19) is universal for two and three dimensions. What about its universality for dimension four? and for the higher ones? I conjectured in [9] that (19) is universal for dimension four, but a stronger result has been proved by Llosa and Soler in [10]. The answer is given by the following

**Theorem 6 (Llosa-Soler).-** *Any $n$-dimensional metric $g$ may be always locally obtained as deformation of a flat metric $\eta$ by a two-form $F$ according to the deformation law* (19):
$$g = \lambda\,\eta + \epsilon\,F^2 \quad .$$

Or, in other words, *this deformation law (19) is universal for any dimension.*

Observe that, by just a redefinition of the scalar function $\lambda(F)$, we can always introduce an additional universal scalar function $\mu$ (i.e. a function that, like $\lambda(F)$, is chosen once for all, irrespective of the particular space-time considered) so as to have
$$g = \lambda\,\eta + \epsilon\,\mu F^2 \quad . \tag{20}$$

Such a scalar $\mu$ may be convenient for geometric or physical reasons (for example, simplification of the expression of $F$ or of the form of its equations).

(c) An example: the Schwarzschild space-time.

Let $\{t, s, \theta, \varphi\}$ be the spherical Minkowski coordinates, of unit velocity $u \equiv dt$, and
$$e \equiv \alpha(s)ds$$
a space-like *radial* vector field. The static, simple and more general spherically symmetric 2-form $F$ is
$$F \equiv u \wedge e = \alpha(s)dt \wedge ds \quad .$$

**Result 1.-** *$F$ is the gravitational potential of Schwarzschild space-time if, and only if,*
$$\alpha = \alpha(U) \quad , \quad U \equiv -m/s \quad .$$

Without loss of generality, because of the introduction of a function $\mu$ in (20), let us take $\alpha(U) = U$. Thus

$$F = -\frac{m}{s}\, dt \wedge ds \tag{21}$$

and one can show:

**Result 2.-** *With the universal deformation law $g = \lambda\,\eta + \epsilon\,\mu F^2$, the gravitational potential $F = -(m/s)\, dt \wedge ds$ generates Schwarzschild metric with*

$$\lambda = 4U^2(1+W)^2 \quad , \quad \mu = \frac{W}{W(1+W)} - 4(1+W)^2 \quad ,$$

*where $W = W(z)$ is the Lambert function and*

$$z \equiv Q\, exp\{-U/2\} \quad .$$

Remember that Lambert function $W(z)$ is defined by $z = W(z) exp W(z)$.

**Result 3.-** *In terms of the Minkowski radial parameter $s$, the universal conformal factor $\lambda$ has the form represented in Figure 7.*

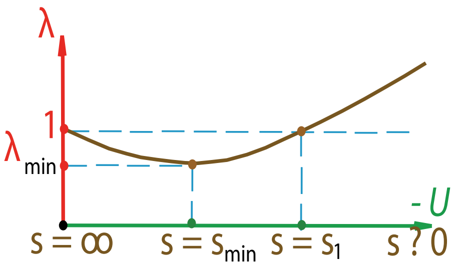


**Figure 7.** The presence of a gravitational field deforms radial distances not monotonically: first contractive up to a value $s = s_{min}$ it is expansive to reach the Minkowskian value at infinity.

But the most interesting result is the following one:

**Result 4.-** *The Minkowskian radial coordinate $s$ and the Schwarzschildian radial coordinate $r$ are related by*

$$r = 2m\,\{1 + W(Q\exp\{s/2m\})\} \tag{22}$$

Equation (22) looks like shows Figure 8. Thus, if we ask how the spherically symmetric

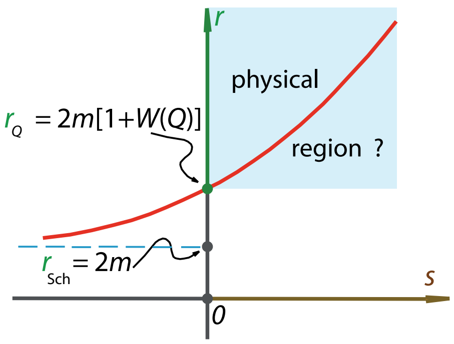


**Figure 8.** A coefficient $Q$ of *gravitational compressibility* naturally appears in the universal law of gravitational deformation which prevent the values *$O \leq r < r_Q = 2m\,\{1 + W(Q)\}$ to be physically reached.*

gravitational potential $F$ given by (21) deforms Minkowski metric (as realized, for example, by means of a field of natural clocks, see below), it appears that a region of Schwarzschild *strictly* containing the standard Schwarzschild radius has no physical meaning. The constant $Q$, which defines how much bigger than the Schwarzschild radius is this value $r_Q$, separating the region of physical meaning in the Schwarzschild solution, plays thus the role of a *gravitational compressibility coefficient.*

*Inertial fields*
In a local, region of the space-time, an inertial observer may consider Minkowski metric $\eta$ as constructed or materialized by means o a system of proper clocks and light signals. We call such a local region his 'laboratory'. Suppose his laboratory 'invaded' by a gravitational field[6]. Our deformation law being universal, this gravitational field $g$, whatever it be, can be described by a gravitational potential $F$, which will *deform* the metric $\eta$, i.e. the behavior of proper clocks and light signals, so as to become $g = \lambda\, \eta + \epsilon\, \mu F^2$.

Conversely, because of the criteria commented above for the choice of two-forms as gravitational potentials, *any* two-form is in principle able to generate a gravitational field $g$, generically curved.

But a restricted class $\{F_I\}$ of two-forms will generate metrics $g = \eta_I$, of vanishing Riemann tensor, i.e. flat metrics, corresponding thus to non-gravitational fields. It seems thus natural to consider these 'internal' two-forms, which deform the flat space-time into itself without creating gravitational fields, as generating *inertial fields.* For this reason, these two-forms are called *inertial potentials.*

Consequently, the differential system for the inertial potential two-forms $F_I$ is:

$$\boxed{\mathrm{Riem}(\lambda(F_I)\, \eta + \epsilon\, \mu(F_I) F_I^2) = 0}$$

Inertial potentials play an important role in the physical interpretation of our universal law of gravitational deformation and of accelerated observers, identified, like in Newtonian theory, as those submitted to inertial fields.

In Minkowski space-time $(\mathbb{R}^4, \eta)$, it is the *structure metric* $\eta$ which defines *inertia*: inertial observers are those such that their unit velocity field $u$ is $\eta$-covariantly constant: $\overset{\eta}{\nabla} u = 0$. Now, Minkowski space-time may be enriched with the class $\{\eta_I\}$ of metrics associated by the universal deformation law to the class $\{F_I\}$ of inertial potentials, $(\mathbb{R}^4, \eta; \{\eta_I\})$, for which the accelerated observers are those such that their unit velocity field $u_I$, $\eta_I(u_I, u_I) = 1$, is $\eta_I$-covariantly constant: $\overset{\eta_I}{\nabla} u_I = 0$.

*Some suggestions*
There remains many open problems and conjectures that we cannot evoke here. Let me finish remarking some points of interest.

1. The n-dimensional proof by Llosa and Soler [10], of my conjecture about the universal character of the gravitational deformation law, gives a conceptual result of great importance, that could have a great influence on our conceptions and uses of relativity and of metric geometries.

From now on, *any local result stated on any arbitrary metric may be reinterpreted as a specific property of some 2-form in a flat background. And vice versa.*

2. We have seen that Schwarzschild metric is generated by a two-form similar to that of the electromgnetic field (up to a power) of a charged point particle. As the deformation law is universal, it would be interesting to study the metrics inspired in other electromagnetic two-forms corresponding to well known configurations. This could help the physical interpretation of many gravitational solutions, specially the static ones. Conversely, the law may be used as a heuristic or phenomenological procedure to find metrics for suitably given gravitational potentials.

[6] This process may be conceived in one of the two forms mentioned in footnote 4.

3. One of the big problems for the quantification of the gravitational field is its invariance under the general pseudo-group of diffeomorphisms, meanwhile quantum theories are only invariant under Poincar group.

In a first step, our law, which associates to some accelerated observers of Minkowski a new flat metric, allow to extend to these accelerated observers the usual methods of quantification.

In a second step, and as to quantify g is equivalent to quantify F in a Minkowski background, the quantification of the gravitational field can be realized under suitable global conditions.

4. The scalars $\lambda(F)$ and $\mu(F)$ in the deformation law $g = \lambda\,\eta + \epsilon\,\mu F^2$ are in fact functions of the scalar invariants of $F$, and these scalar invariants are even monomials of $F$. Consequently, it is only the *absolute value* $|F|$ of $F$, $|F| \equiv \{F, -F\}$, which participates in the metric $g$. A pertinent question thus appears: what spin corresponds to such an absolute value $|F|$ of a 2-form? As the metric field g is now a consequence of the sole presence of the field $|F|$, this question on the spin of the graviton is of great interest.

5. The universal law of gravitational deformation could throw new light on the notion of gravitational energy, which now appears related to the gravitational potential $|F|$.

6. At present, *all* applied models of gravitational fields are obtained from a development of the metric of the form

$$g = \eta + h \tag{23}$$

which is certaintly *mathematically* valid, but is it *physically* valid? At the light of the universal deformation law, the metric tensor $\eta$ in (23) would not be a good physical choice but at the first order, so that the term in h would present useless complications without physical meaning. A development in $F$, $\lambda$ and $\mu$ around the form

$$g = \lambda\,\eta + \epsilon\,\mu F^2 \tag{24}$$

could avoid these features. But to develop it one needs the complete determination of the universal scalars $\lambda$ and $\mu$ in terms of the two scalar invariants of F. I am now working on this subject.

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