INSTITUTE OF PHYSICS PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **23** (2006) 7395–7416 doi:10.1088/0264-9381/23/24/012

Relativistic positioning systems: the emission coordinates

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Received 28 June 2006, in final form 2 October 2006
Published 8 November 2006
Online at stacks.iop.org/CQG/23/7395

Abstract

This paper introduces some general properties of the gravitational metric and the natural basis of vectors and covectors in four-dimensional emission coordinates. Emission coordinates are a class of spacetime coordinates defined and generated by four emitters (satellites) broadcasting their proper time by means of electromagnetic signals. They are a constitutive ingredient of the simplest conceivable relativistic positioning systems. Their study is aimed to develop a theory of these positioning systems, based on the framework and concepts of general relativity, as opposed to introducing 'relativistic effects' in a classical framework. In particular, we characterize the causal character of the coordinate vectors, covectors and 2-planes, which are of an unusual type. We obtain the inequality conditions for the contravariant metric to be Lorentzian, and the non-trivial and unexpected identities satisfied by the angles formed by each pair of natural vectors. We also prove that the metric can be naturally split in such a way that there appear two parameters (scalar functions) dependent exclusively on the trajectory of the emitters, hence independent of the broadcasted time, and four parameters, one for each emitter, scaling linearly with the time broadcasted by the corresponding satellite, hence independent of the others.

PACS numbers: 04.20.−q, 95.10.Jk

1. Introduction

A *relativistic positioning system* is basically constituted by four *emitters*, i.e. by four clocks broadcasting a time scale τ^A $(A = 1, \ldots, 4)$ by means of electromagnetic signals[3]. In

[3] A time scale is a rhythm able to be generated by the repetition of a unit, together with the assignation of a origin.

particular, the time scale of every clock may be based on its *proper time*. This is what, for simplicity, is supposed here.

The spacetime events on the future-oriented null cones with vertices on every emitter worldline are then *physically labelled* by the value of the time scale at the vertex, and so, define four (one-parameter) families of null hypersurfaces, say $\tau^A =$ constant. Generically, these families constitute the coordinate hypersurfaces of a coordinate system. The physical coordinates so generated are called *emission coordinates*. This is a sound name, since in this construction the coordinates themselves are broadcasted by the four emitter worldlines.

The interest of emission coordinates lies in that they are constitutive ingredients of (relativistic) *positioning systems*. What, in turn, is the interest of these systems?

1.1. Relativistic approach to global navigation systems, space physics and solar system astronomy

In a fully relativistic theory of *location systems,* that is in a theory entirely based on the framework and concepts of general relativity and concerning the physical construction of coordinate systems, relativistic positioning systems appear as the *best* systems that today we are able to conceive (see for example [1]).

For the physical analysis of solar systems, general relativity may be either just applied as a learned algorithm to sprinkle Newtonian expressions with corrective 'relativistic effects' (post-Newtonian perturbation methods) or considered as providing the best concepts on spacetime and gravitation.

But today we know that the physical model of spacetime and gravitation offered by general relativity is better adapted to nature than the model offered by Newtonian theory. As a consequence, Newtonian analysis of global navigation systems, space physics or solar system astronomy, for example, needs more and more relativistic corrections.

In this situation, the interest of a complete relativistic approach to physical problems, naturally integrating in their starting quantities all the relativistic corrections, without making them explicit as perturbations with respect to an insufficient (and incorrect!) Newtonian theory, seems evident[4]. Aged almost a century, it is time to consider the infancy of relativity, paternally guided by Newtonian theory, as finished. It is obvious that it is not in taking refuge in out-of-date Newtonian concepts that one will be able, where relativity itself is concerned, to ask vanguard scientific questions. We believe that general relativity theory is ripe enough, and that the present moment is technically and scientifically interesting enough, to develop general relativity in an adult, autonomous form, without reference to Newtonian theory[5].

[4] In fact, many relativistic analyses and descriptions of physical systems, including basic properties of the electromagnetic field, shock waves, hydrodynamics and magnetohydrodynamics, detonation and deflagration waves, are much more easily done in relativity than in Newtonian theory. Contrary to an extended opinion, they are the high precision quantitative developments searched for in some physical systems which make relativistic calculations long and complex, meanwhile the starting relativistic descriptions of these same physical systems remain relatively simple. Only in the numerical computation of these high precision results, but not in the conceptual context of defining and analysing the corresponding physical systems, could Newtonian theory be accepted today as the zeroth order term of a numerical algorithm.

[5] We believe that relativity has not still revealed all its capacities for amazing us. Nevertheless, apart from the structural or mathematical generalizations that it suggests, its heuristic power is at present deflated by the oppressive presence of Newtonian theory in almost all its applications. We hope that an autonomous development of general relativity for physical applications, starting from its proper basis and excluding any *a priori* help of Newtonian conceptions, may reveal new specific features of conceptual, scientific and technical interest. First results in this direction are those already proposed elsewhere and commented below on the new, paradigmatic ways of using global navigation satellite systems or of constructing emission coordinates for the solar system .

But the task is not easy. The first obstruction to such an autonomous development of general relativity is the almost absolute lack of relativistic operational protocols for constructing coordinate systems. On one hand, this lack is due to the fact that in Newtonian physics such protocols are supposed well known and trivial and that, as mentioned above, relativity is frequently used to find relativistic corrections to the Newtonian picture of the physical system in question. On the other hand, it is also due to a misunderstanding of the meaning of the relativistic covariance principle, frequently erroneously interpreted as stating the lack of physical interest of coordinate systems[6].

This is why the first step for an autonomous relativistic analysis of physical situations is the elaboration of a fully relativistic theory of physical coordinate systems or location systems, whose basic ingredients have been obtained elsewhere (see for example [1] or [2]). Of these relativistic location systems, the aforementioned class of *positioning systems* are the sole which are *immediate*, i.e. that are able to indicate to every event its proper coordinates *without delay* [1, 2]. It is this important property that allows us to say that emission coordinates are the best systems conceived up until today.

It is the heuristic character of such an autonomous relativistic approach that has already suggested not only the notion itself of emission coordinates but also their two important applications: the use of global navigation satellite systems as principal reference systems for the Earth (project SYPOR, see [1]) and the use of millisecond pulsar signals as emission coordinates for the solar system and its neighbours (see [3]).

1.2. Null coframes

In the study of emission coordinates, a first step is the analysis of the differential form of their intrinsic properties, the most natural form to incorporate them into the standard differential geometric frame of general relativity. Generically, these differential properties are nothing but those inherited by the natural basis that emission coordinates induce on the tangent and cotangent spaces at every event or, in other words, those of the *emission frames* and *emission coframes* associated with emission coordinates.

Because the coordinate hypersurfaces $\tau^A =$ constant of emission coordinates are light cones, the four 1-forms $\mathrm{d}\tau^A$, defining the natural or coordinate coframe[7] $\{\mathrm{d}\tau^A\}$ of the coordinate system $\{\tau^A\}$, are all null. Such a coframe of four future-oriented null covectors is of a very unusual causal class[8] and will be called an *emission coframe*.

To an emission coframe corresponds by metric duality to the frame of null vectors $(\vec{\ell}^A)^\mu = g^{\mu\nu}(\mathrm{d}\tau^A)_\nu$ determining the null geodesics along which the signals propagate. Note that neither these vectors nor their directions coincide with any of the vectors or the directions

[6] The covariance principle is one of the general principles in physics that helps us to better understand the internal structure of physical laws, allowing us to express them in an easy and clear form. It is an extension of the other one of dimensional invariance, which states that the physical laws are independent of the particular units used to obtain them. But, can we infer from these principles that the tasks of definition and construction of coordinate systems or units are not of physical interest? Certainly not. The importance of these principles lies in what they allow *separating* the difficulties inherent to the conception and construction of every experiment and to the control of the context in which it takes place, from the difficulties inherent to the conception and construction of the coordinate systems for its location and of the units for the measure of its specific quantities. It would be a dangerous nonsense for physics to believe that, instead of simply allowing this separation, these principles of covariance and of dimensional invariance scorn the physical construction of coordinate systems and units. Far from that, these principles reinforce the need to *first* improve the construction of coordinate systems and units as one efficient way to *afterward* improve physical laws.

[7] As usual, the word *coframe* is used as a shortcut for a frame of covectors.

[8] For the classification of all the 199 causal classes of frames of the spacetime see [4]. For some coordinate examples see [5].

of the reciprocal basis: $\vec{\ell}^A \neq \partial_A$. Indeed, as we will see, the natural or coordinate vectors, ∂_A, of these coordinates are not null but all spacelike.

Let us observe that the appellation *null coframe* is also very appropriate for an emission frame $\{d\tau^A\}$, because describing the causal orientation of *all* its real constituents, the covectors $d\tau^A$. Nevertheless, attention must be paid to the context in which it is used, because this appellation has also been frequently applied to the very different Newman–Penrose frames [6], both in their real and complex versions.

In fact, this last usage has induced some misconceptions about the existence of real frames of real null vectors or covectors in a spacetime of hyperbolic signature. The point is that for Newman–Penrose frames the real pair of null vectors is chosen orthogonal to the remaining vectors of the frame, preventing these last ones to be also real and null. And it is this same orthogonality choice which imposes that the dual frame contains similarly a pair of real null vectors. The absence of this orthogonality constraint in emission frames, because of its absence of physical role, allows these frames, on one hand, to be constituted by four real null covectors, and forbids their dual frames, on the other hand, to contain real null vectors, as commented on above.

Real null frames seem to have been first considered by Zeeman [7] as a device for a technical proof. Derrick [8] discovered a class of them as particular *symmetric frames*, later studied by Coll and Morales [9], who also proved that real null frames constitute a causal class among the 199 possible ones [4]. Coll [10] seems to have been the first to construct physical coordinate systems by means of light beams. The real null frames associated with light beams are, in some sense, dual to emission frames: they are the natural vectors which are null in this case, while the covectors are spacelike. Symmetric real null frames have also been proposed by Finkelstein and Gibbs [11] as a checkerboard lattice for a quantum spacetime. It is also Coll [12–14] who seems to have been the first to propose the physical construction of relativistic coordinate systems by means of broadcast light signals, whose natural coframe is an emission frame. Bahder [15] has obtained explicit calculations for the vicinity of the Earth at first order in the Schwarzschild spacetime, and Rovelli [16] has developed a particular case where emitters define a symmetric frame in Minkowski spacetime, as representative of a complete set of gauge invariant observables. Blagojević *et al* [17] analysed and developed the symmetric frame considered in Finkelstein and Rovelli papers. Recently Lachièze-Rey [18] has considered applications of emission coordinates to cosmology and positioning. Some more specific papers on positioning systems have been published [1–3, 19–23], an international school has been devoted to the subject [24], and some works are in progress [25–28].

Many of these references show that they are aware of the need for physical coordinate systems (location systems) in experimental projects involving relativity, but the importance of their role depends on the authors. For example, Bahder [15], who works in the vicinity of the Earth at first order in Schwarzschild spacetime, considers emission coordinates as a simple intermediate way to allow users to know their location with respect to another, exterior, previously given and not immediate coordinate system (WGS84), meanwhile our point of view is that emission coordinates are those that are assigned to be the *primary* systems of any *relativistic cartography*.

To complete this bibliographic account, let us comment on *GPS coordinates*, occasionally used as an alternative name for emission coordinates. The fact that in emission coordinates the location of events is obtained from the signals broadcasted by four emitters (satellites) strongly relates them to the global positioning system, the most high-performance GNSS at present. This is why some authors, in a first contact with the subject, have given them the name of GPS coordinates or coordinates of GPS type. One of us seems to have been the first [12] to use this appellation, but other authors have also used it in [16], in [17] and recently in [18]. Nevertheless, at present we are not in favour of this apellation. First of all, because it is a confusing one: the name 'GPS coordinates' is already very largely used to mention something completely different to emission coordinates, namely to mention the conventional coordinates, whatever they be, when they are obtained by means of GPS receivers. But also because the GPS has never considered the broadcasted signals as coordinates by themselves, but only as information able to find basically the coordinates in the World Geodetic System (WGS84), a conventional three-dimensional coordinate system on the Earth's surface, in no way similar to emission coordinates.

1.3. About the present results

The paper is devoted to the properties of emission frames[9]. The null and future directed character of the covectors imply a particular form for the contravariant metric: the diagonal components vanish, $g^{AA} = 0$, the extra-diagonal terms are positive, $g^{AB} > 0$, and the three geometric means $\sqrt{g^{12}g^{34}}$, $\sqrt{g^{13}g^{24}}$ and $\sqrt{g^{14}g^{23}}$ satisfy the inequalities for the sides of a triangle. Indeed, the determinant of the metric depends only on these three quantities. The natural vectors are all spacelike. Moreover, the coordinate 2-surfaces are spacelike. This means that each pair of vectors, ∂_A, ∂_B, of an emission frame defines a spacelike plane and forms an angle θ_{AB} between them. We deduce that the six angles so obtained satisfy four identities: $\theta_{12} = \theta_{34}, \theta_{13} = \theta_{24}, \theta_{14} = \theta_{23}$ and $\theta_{12} + \theta_{23} + \theta_{13} = 2\pi$, thus they have only two degrees of freedom. These results involve the characterization of the covariant and contravariant gravitational metric in emission coordinates and lead us to a natural, physically meaningful, normalization of emission frames. In turn, this results into an interesting splitting of the six degrees of freedom of the metric into two clearly different types of parameters: Four *scaling parameters*, each corresponding to each emitter, dependent linearly on the proper time transfer of its corresponding clock and independent of the other clocks. And two *scale-invariant parameters*, corresponding to the two degrees of freedom of the angles θ_{AB}, depending exclusively on the relative directions of the clocks, determined by the worldline of the emitters, independently of the frequency of the emitted series of signals generated by the clocks. This last result has been presented without proof in [20, 21]. For comparison, we present the analogous splitting for three-dimensional spacetimes. In this case the three degrees of freedom of the metric split into three scaling parameters and no scale-invariant parameter.

The properties of emission frames are obtained by using known geometric results for spacetimes, that is for four-dimensional metric spaces of Lorentzian signature. Thus, our presentation stresses the geometry of the problem by using, when possible, intuitive (and rigorous) deductions rather than algebra. Nevertheless, some of these properties are afterwards algebraically checked.

The properties obtained here will be used in a next work to analyse the precise conditions for continuity, differentiability and regularity of emission coordinates. They are also unavoidable in order to tackle local and global features of emission coordinates and their relationship with more usual coordinates. Elsewhere we will present [26, 27] the properties obtained for static and stationary positioning systems in flat Minkowski spacetime.

The paper is organized as follows. In section 2 we define emission coordinates and describe the physical situation in which they appear. In section 3 we study the natural coframe and the corresponding contravariant metric. Then, in section 4 we study their counterpart: the natural frame and the corresponding covariant metric. The properties obtained lead to the above-mentioned splitting of the metric, presented in section 5. In section 6 we present the

[9] As the frames of vectors and covectors are biunivocally related, we will refer to them generically as 'emission frames'.

relationship between the properties obtained for the covariant and the contravariant forms of the metric, and in section 7 we conclude with a brief analysis of the results obtained and a sketch of the work in progress. Finally, in the appendix, we describe a possible algorithm, allowing to construct emission coordinates in any space time and for any set of four emitters.

2. Description of the emission coordinates

To provide a region of the spacetime with a system of coordinates, it is sufficient to give four scalar fields, say x^μ. These fields must satisfy some sufficient conditions of smoothness and regularity: each of them need to be continuous and differentiable until some degree, and their set has to be non-degenerate, i.e. functionally independent: $\mathrm{d}x^1 \wedge \cdots \wedge \mathrm{d}x^4 \neq 0$. Then, their equipotential hypersurfaces $x^\mu =$ constant, considered as (one-parameter families of) coordinate hypersurfaces, define unambiguously the coordinate system.

If the four scalar fields are realized as direct measures of physical fields, we say that they constitute a location system or a system of physical coordinates [1].

As physical fields, the four scalar fields have to be easy to be produced, regular in their propagation, sufficiently weak to be considered as *passive*[10] and sufficiently strong to be measured by a local receiver in the spacetime domain in question.

For many applications in space physics and Earth sciences, the best candidates for these scalar fields are electromagnetic signals, and the simplest way to parametrize these signals is to submit them to broadcast some time scales, as those based on the proper time τ^A of natural clocks (emitters).

In fact, as already said, the simplest representative of positioning systems is a system constituted by four clocks broadcasting their proper time scales by means of electromagnetic signals[11].

The physical situations supposed here are such that every emitter (i.e. clock with emission mechanism of electromagnetic signals broadcasting its proper time) may be considered as point-like, so that the history of every emitter in the spacetime is described by a timelike worldline.

Also, the broadcasting electromagnetic signals are supposed correctly described by ideal electromagnetic signals propagating along null geodesics in vacuum (electromagnetic optical approximation). Hence, the instants of an emitter are vertices of future null cones that foliate the space time.

Each null cone is labelled by the time of its emission (vertex) and, by interpolation, they define a scalar field. Then, four different emitters define four scalar fields, which can constitute a coordinate system for some region of the spacetime (see figure 1). These are our emission coordinates.

In emission coordinates, the coordinates of an arbitrary event are the four times $\{\tau^A\}$ of emission, carried by the four null cones of the system, received at the event.

An alternative and complementary view of this construction, important because it directly induces a simpler way to compute the emission coordinates of an event, is based on the sole past null cone of the event. This past null cone contains every incoming null geodesic, in particular, the four geodesics followed by the signals reaching the event from the four emitters. The intersection of this cone with each of the four worldlines determines the emission coordinates of the event (figure 1).

[10] Here a scalar field is said to be passive if its interaction with the events to be located may be considered as negligible. See [1] or [12] for related properties of a physical coordinate system.

[11] The time scales of the clocks of positioning systems, be they based on proper time or not, are independent, i.e. not constrained to be synchronized in any way.

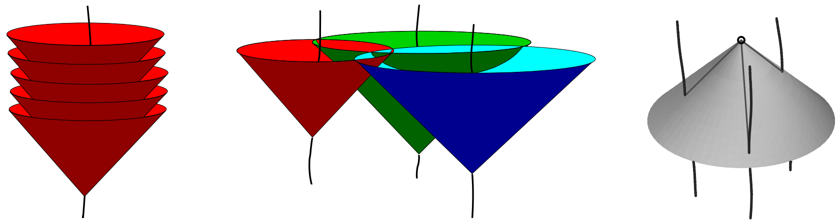

Figure 1. Representation of three emitters in a three-dimensional spacetime (time upwards). The lines are the spacetime trajectories (worldlines) of the emitters. The left figure shows the foliation defined by the spacetime surfaces (future light cones) visited by each value of the signal series of one emitter. The middle figure represents three coordinate surfaces of the emission coordinates defined by the series of three emitters. The right figure gives a complementary view: the intersection of the past light cone of an event with the trajectories of the emitters gives the emission coordinates of this event.

3. The natural covectors and the contravariant metric in emission coordinates

3.1. Causal character of the natural covectors

As already said, the coordinate hypersurfaces $\tau^A =$ constant of emission coordinates are the future null cones with the vertex at the instant τ^A of the source A. Let us enhance the first immediate property, already mentioned.

Property 1. *The coordinate covectors of emission coordinates,* $\mathrm{d}\tau^A$*, are null.*

Time scales being naturally future increasing, of the hypersurfaces $\tau^A = t_1$ and $\tau^A = t_2$, with $t_2 > t_1$, the later one is a null cone emitted after the former: the $\tau^A = t_2$ is in the causal future of $\tau^A = t_1$. This implies that any future-oriented timelike curve, $C(l)$, will cross the coordinate hypersurfaces $\tau^A =$ constant in increasing order: $\frac{dC^\mu}{\mathrm{d}l}\partial_\mu\tau^A > 0$. Equivalently, for all timelike and future-directed vector u , the application of the 1-form $\mathrm{d}\tau^A$ is positive: $\mathrm{d}\tau^A(u) > 0$. That is,

Property 2. *The coordinate covectors of emission coordinates,* $\mathrm{d}\tau^A$*, are future directed.*

These two properties implies that the metric product of every one of these covectors by itself is null: $\mathrm{d}\tau^A \cdot \mathrm{d}\tau^A = 0$, and that, for signature $(+---)$, the crossed metric products are positive: $\mathrm{d}\tau^A \cdot \mathrm{d}\tau^B > 0, \forall A, B \neq$. Therefore, the contravariant metric in emission coordinates has the form

$$(g^{AB}) = \begin{pmatrix} 0 & g^{12} & g^{13} & g^{14} \\ g^{12} & 0 & g^{23} & g^{24} \\ g^{13} & g^{23} & 0 & g^{34} \\ g^{14} & g^{24} & g^{34} & 0 \end{pmatrix}, \qquad (1)$$

where $g^{AB} > 0$ for $A \neq B$.

3.2. Linear independence of the covectors

Let us denote the natural covectors by $\ell^A \equiv \mathrm{d}\tau^A$. For emission coordinates in a tridimensional spacetime, the inequality $g^{AB} > 0$ would suffice to guarantee the linear independence of the cobasis $\{\ell^A\}$. In contrast, for the four-dimensional case, this inequality is not sufficient. This difference can be expressed with the following two facts, valid for Lorentzian spacetimes.

- Three null vectors are linearly dependent if and only if two of the vectors are proportional. That is, three *different* null directions always span a tridimensional spacetime.
- Four null vectors, none of them proportional to another, can be linearly dependent. That is, four *different* null directions do not necessarily span a four-dimensional spacetime.

This is easy to see since there are only two null directions in a bidimensional spacetime, but there are infinite null directions in a tridimensional spacetime. Thus, we can always choose four null directions living in any tridimensional timelike subspace of the four-dimensional spacetime. The linear dependence of the four covectors $\{\ell^A\}$ is reflected in the vanishing of the determinant of the metric (1): $|g^{AB}| = 0$.

3.3. Lorentzian signature of the metric

The condition $|g^{AB}| \neq 0$ is required in order to have a regular or non-degenerate metric. However this condition does not ensure that the metric is of Lorentzian signature, which is required by the spacetime properties. This requirement introduce another qualitative difference between the tridimensional and the four-dimensional case.

- The existence of null covectors forbids the Euclidean signatures in both dimensions: (+++), (−−−), (++++) and (−−−−).
- The Lorentzian signatures (−++) and (−+++), in three and four dimensions respectively, are forbidden by the condition $\ell^A \cdot \ell^B > 0$, since, in this case, it implies that ℓ^A has opposite time orientation to ℓ^B, being impossible to have three covectors, ℓ^1, ℓ^2, ℓ^3, with mutually opposite orientation.

Therefore,

- three null covectors, $\{\ell^A\}$, with positive scalar product for each pair, $\ell^A \cdot \ell^B > 0$ $(A \neq B)$, span a tridimensional space of Lorentzian signature (+−−);
- four *linearly independent* null covectors, $\{\ell^A\}$, with positive scalar product for each pair, $\ell^A \cdot \ell^B > 0$ $(A \neq B)$, span a four-dimensional space of either Lorentzian signature (+−−−) or null signature (++−−).

This means that, in three dimensions, the Lorentzian signature (+−−) of g^{AB} is automatically satisfied. This can be checked by observing that the determinant of the metric is positive:

$$\det(g^{AB}) = \begin{vmatrix} 0 & g^{12} & g^{13} \\ g^{12} & 0 & g^{23} \\ g^{13} & g^{23} & 0 \end{vmatrix} = 2g^{12}g^{13}g^{23} > 0.$$

However, in four dimensions, to select only the Lorentzian signature, (+−−−), we need to impose the additional condition $\det(g^{AB}) < 0$, which, from the above discussion, is necessary and sufficient.

Let us compute the determinant of the metric. After some manipulation a very interesting factorization of the determinant can be obtained:

$$|g^{AB}| = \begin{vmatrix} 0 & g^{12} & g^{13} & g^{14} \\ g^{12} & 0 & g^{23} & g^{24} \\ g^{13} & g^{23} & 0 & g^{34} \\ g^{14} & g^{24} & g^{34} & 0 \end{vmatrix} = A^4 + B^4 + C^4 - 2A^2B^2 - 2A^2C^2 - 2B^2C^2$$
$$= (A + B + C)(A - B - C)(B - A - C)(C - A - B), \quad (2)$$

where A, B, C are the geometric means of the three products of complementary components:

$$A \equiv \sqrt{g^{23}g^{14}}, \qquad B \equiv \sqrt{g^{13}g^{24}}, \qquad C \equiv \sqrt{g^{12}g^{34}}. \quad (3)$$

Thus, these three positive parameters, $A, B, C > 0$, are the appropriate ones to express the conditions for the metric to have non-degenerate Lorentzian signature:

$$\det(g^{AB}) < 0 \iff (B + C - A)(A + C - B)(A + B - C) > 0$$
$$\iff A < B + C, \quad B < A + C \quad \text{and} \quad C < A + C. \qquad (4)$$

That is,

Theorem 1. *The metric of the form (1) is a non-degenerate Lorentzian metric if and only if the geometric means A, B, C defined by (3) can be the lengths of the sides of a triangle (4).*

This image is more evident if we realize that the determinant of the metric is minus the Heron polynomial:

$$-\det(g^{AB}) = \mathcal{H}(A, B, C) \equiv (A + B + C)(B + C - A)(A + C - B)(A + B - C). \qquad (5)$$

Let us recall that the Heron polynomial gives the area of a triangle in terms of the length of its sides A, B, C:

$$\text{Area}\ {}^{A}\underset{C}{\triangle}{}^{B} = \tfrac{1}{4}\sqrt{\mathcal{H}(A, B, C)}.$$

4. The natural vectors and the covariant metric in emission coordinates

From the contravariant metric, g^{AB}, the covariant metric, g_{AB}, is obtained by inverting the matrix of their components: $(g_{AB}) = (g^{AB})^{-1}$. However this method does not make the geometric properties of the covariant metric evident which, as for the contravariant metric, are consequence of the Lorentzian signature and of the null and future-directed nature of the covectors $\mathrm{d}\tau^A$. For this reason we are going to derive the properties of the covariant metric in emission coordinates by some simple geometric deductions from well-known properties of Lorentzian spaces. This procedure leads to a natural splitting of the covariant metric. In the next section we will obtain the relationship between the ingredients of both metrics by the inversion of one of them.

4.1. Causal character of the natural vectors

The natural covectors, $\mathrm{d}\tau^A$, of emission coordinates are null. However, the natural vectors, ∂_{τ^A}, of the dual frame do not have this character (except for bidimensional spacetimes). The causal character of the covector $\mathrm{d}\tau^A$ is given by the nature of the coordinate hypersurface $\tau^A =$ constant. In contrast, the causal character of the vector ∂_{τ^A} is given by the nature of the coordinate line $\tau^B =$ constant $\forall B \neq A$. The other object that, from the causal point of view, characterizes the coordinates is the causal nature of the coordinate 2-surfaces (see [4]), which can be represented by the bivectors $\partial_{\tau^A} \wedge \partial_{\tau^B}$.

The coordinate lines of emission coordinates are the intersection of three coordinate hypersurfaces (light cones). In a null hypersurface, all the directions are spacelike except one which is null. Hence, the coordinate lines must be either spacelike or null. In addition, at any point of the null hypersurface, the tangent hyperplane is determined by the null direction. Thus, any two null hypersurfaces containing the same null direction at some common point are tangent at this point. Therefore, the coordinate line can be null only where the three hypersurfaces are tangent, i.e. where the coordinate system is degenerate. Consequently,

Property 3. *The coordinate vectors ∂_{τ^A} of emission coordinates are spacelike.*

An alternative way to arrive at this conclusion is the following. Let us consider the null metric dual vectors $\vec{\ell}^A$ of the null covectors ℓ^A, $(\vec{\ell}^A)^\mu = g^{\mu\nu}(\ell^A)_\nu$. Each coordinate vector, $s_A \equiv \partial_{\tau^A}$, is perpendicular to the 3-space spanned by the complementary vectors in the reciprocal basis:[12]

$$s_A \perp \mathcal{S}_A \equiv \mathrm{span}(\vec{\ell}^B, \vec{\ell}^C, \vec{\ell}^D), \qquad \text{with} \quad A, B, C, D \neq .$$

The 3-space $\mathcal{S}_A$ is timelike since, by definition, it contains more than one null direction. Hence, the orthogonal vector s_A is spacelike.

This type of argument gives also the causal character of the coordinate bivectors, $s_A \wedge s_B$, that is, the causal character of the plane spanned by each pair of vectors s_A, s_B. This plane is completely orthogonal to the plane spanned by the dual vectors $\vec{\ell}^C, \vec{\ell}^D$, with $A, B, C, D \neq$. This latter is timelike, since it contains two null directions. Therefore,

Property 4. *The planes spanned by each pair of coordinate vectors of emission coordinates, $s_A \wedge s_B$, are spacelike.*

The 3-space spanned by each triad of vectors,

$$\mathcal{S}^D \equiv \mathrm{span}(s_A, s_B, s_C), \qquad \text{with} \quad A, B, C, D \neq,$$

is orthogonal to the complementary dual vector $\vec{\ell}^D$, which is null. Hence, although the three vectors are spacelike, they span a null space.

Property 5. *The 3-spaces spanned by each triad of coordinate vectors of emission coordinates, $s_A \wedge s_B \wedge s_C$, are null. Their unique null direction is given by the vector $\vec{\ell}^D$ $(A, B, C, D \neq)$.*

4.2. The angles between the natural vectors

The space $\mathcal{S}^D$ is null, so that any metric quantity will not depend on the null direction $\vec{\ell}^D$. We can then consider its quotient space by the null direction, $\mathcal{S}^D/\vec{\ell}^D$, which is a bidimensional space with an (anti)-Euclidean induced metric. Let us denote the equivalence class of the vector $v \in \mathcal{S}^D$ by $\langle v \rangle \in \mathcal{S}^D/\vec{\ell}^D$. The *non-oriented* angle formed by two vectors is the positive angle given by the metric:

$$s_A \cdot s_B = -|s_A||s_B| \cos\theta_{AB}, \qquad \text{with} \quad 0 \leqslant \theta_{AB} \leqslant \pi.$$

Note that this last condition implies $\theta_{AB} = \theta_{BA}$. The same angle is also well defined for the equivalence classes, so that θ_{AB} is also the angle between $\langle s_A \rangle, \langle s_B \rangle \in \mathcal{S}^D/\vec{\ell}^D$. But the three vectors $\langle s_A \rangle, \langle s_B \rangle, \langle s_C \rangle$ are in a bidimensional Euclidian space. This provides the following.

Lemma 1. *The non-oriented angles, $\theta_{AB}, \theta_{BC}, \theta_{AC}$, formed by three spacelike vectors s_A, s_B, s_C in a null tridimensional space, $\mathcal{S}^D$, satisfy either*

$$\theta_{AB} + \theta_{BC} + \theta_{CA} = 2\pi \qquad \textit{or} \qquad \theta_{AB} + \theta_{BC} = \theta_{CA}$$

for some angle, θ_{CA}, of them.

The different possibilities depend on the disposition of the vectors in the plane (see figure 2).

The null vector $\vec{\ell}^D$ belongs to the space $\mathcal{S}^D$ so that it can be spanned by the three vectors:

$$\vec{\ell}^D = \alpha s_A + \beta s_B + \gamma s_C \qquad \text{for some} \quad \alpha, \beta, \gamma \in \mathbb{R}. \tag{6}$$

[12] After a set of objects, the symbol $\neq$ means here that the objects are pair-wise different.

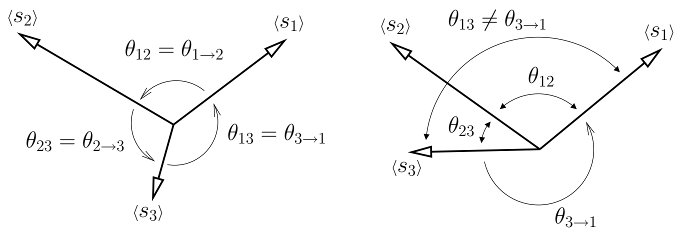

Figure 2. Angles between three vectors in an Euclidian bidimensional space. If the angles are oriented then the sum of the three angles is a complete revolution ($\theta_{1\to 2} + \theta_{2\to 3} + \theta_{3\to 1} = 2\pi$). But if they are non-oriented ($0 \leqslant \theta_{AB} \leqslant \pi$), this is only true if the origin is inside the triangle formed by the end points of the vectors. In any other case there will be an oriented angle bigger than π, say $\theta_{3\to 1} > \pi$, so that when considering non-oriented angles the identity satisfied will be $\theta_{13} = \theta_{12} + \theta_{23}$.

For an emission tetrad the four null vectors $\vec{\ell}^D$ are future oriented. This implies that the three coefficients are positive, for instance

$$\ell^A(\vec{\ell}^D) = \alpha > 0.$$

Taking now the equivalence classes of equation (6), we obtain

$$0 = \alpha\langle s_A\rangle + \beta\langle s_B\rangle + \gamma\langle s_C\rangle, \qquad \text{with} \quad \alpha, \beta, \gamma > 0,$$

which implies that the origin is inside the triangle with vertices at the end of the three vectors. Thus, the future orientation selects one of the possibilities of lemma 1.

Lemma 2. *Let $\{\ell^A\}$ be an emission tetrad and $\{s_A\}$ its reciprocal basis. The non-oriented angles, $\theta_{AB}, \theta_{BC}, \theta_{AC}$, formed by any triad of vectors s_A, s_B, s_C of the reciprocal basis, satisfy*

$$\theta_{AB} + \theta_{BC} + \theta_{CA} = 2\pi.$$

Taking this identity for the four possible triads of vectors, we obtain another identity:

$$\left.\begin{array}{l} +(\theta_{AB} + \theta_{BC} + \theta_{AC} = 2\pi) \\ +(\theta_{AB} + \theta_{BD} + \theta_{AD} = 2\pi) \\ -(\theta_{AC} + \theta_{CD} + \theta_{AD} = 2\pi) \\ -(\theta_{BC} + \theta_{CD} + \theta_{BD} = 2\pi) \end{array}\right\} \quad \Rightarrow \quad \theta_{AB} = \theta_{CD},$$

for any $A, B, C, D \neq$. Thus the angle formed by any pair of vectors, s_A, s_B, coincides with the angle formed by the complementary pair, s_C, s_D.

Theorem 2. *Let $\{\ell^A\}$ be an arbitrary emission tetrad and $\{s_A\}$ its reciprocal basis. The non-oriented angles formed by the spacelike vectors s_A satisfy*

$$\theta_{12} = \theta_{34}, \qquad \theta_{13} = \theta_{24}, \qquad \theta_{23} = \theta_{14},$$

and

$$\theta_{12} + \theta_{13} + \theta_{23} = 2\pi, \qquad 0 < \theta_{12}, \theta_{13}, \theta_{23} < \pi. \tag{7}$$

Let us observe that this coincidence of angles by pairs is not a trivial condition. These results say that for whatever four linearly independent null wave-fronts:

- The intersection of three wave-fronts is a spacelike line. This gives the causal nature of the four dual vectors.

- The intersection of two wave-fronts is a spacelike surface containing two of these lines. This gives the causal nature of the six planes formed by each pair of dual vectors.
- The angle formed by any two of the lines coincide with the angle formed by the other two lines.
- The sum of the three angles formed by one line and each of the other three lines is 2π.

4.3. The length of the natural vectors

Observe that the last result says that there are only two degrees of freedom for the six angles. The other four degrees of freedom of the metric are contained in the *lengths* of the four vectors s_A,

$$\mu_A \equiv \sqrt{-s_A \cdot s_A},$$

which are independent parameters. These four lengths, μ_A, and the angles, θ_{AB}, (six independent parameters in total) determine the covariant metric:

$$g_{AA} = -{\mu_A}^2$$

$$g_{AB} = -\mu_A \mu_B \cos\theta_{AB} \qquad \text{for} \quad A \neq B.$$

Thus,

$$(g_{AB}) = -\begin{pmatrix} {\mu_1}^2 & \mu_1\mu_2 Z & \mu_1\mu_3 Y & \mu_1\mu_4 X \\ \mu_1\mu_2 Z & {\mu_2}^2 & \mu_2\mu_3 X & \mu_2\mu_4 Y \\ \mu_1\mu_3 Y & \mu_2\mu_3 X & {\mu_3}^2 & \mu_3\mu_4 Z \\ \mu_1\mu_4 X & \mu_2\mu_4 Y & \mu_3\mu_4 Z & {\mu_4}^2 \end{pmatrix},$$

where

$$X \equiv \cos\theta_{23}, \qquad Y \equiv \cos\theta_{13}, \qquad Z \equiv \cos\theta_{12}, \tag{8}$$

and these three angles must satisfy the constraint (7), or its equivalent form

$$X^2 + Y^2 + Z^2 - 2XYZ = 1 \qquad \text{and} \qquad \begin{cases} YZ > X \\ XZ > Y \\ XY > Z \end{cases} \tag{9}$$

which can be checked (with some effort) to imply the existence of the angles: $X^2, Y^2, Z^2 < 1$, and their constraint (7).

5. The normalized emission frame and the splitting of the metric

5.1. The normalized emission frame

We have seen that all the vectors of the reciprocal basis of an emission frame are spacelike and that their lengths are independent metric parameters. This leads to defining the *normalized vectors*:

$$\hat{s}_A \equiv s_A/\mu_A. \tag{10}$$

They constitute a normalized basis with the *normalized metric* $\hat{g}_{AB}$:

$$\hat{s}_A \cdot \hat{s}_B \equiv \hat{g}_{AB}, \qquad \text{where} \quad (\hat{g}_{AB}) = -\begin{pmatrix} 1 & Z & Y & X \\ Z & 1 & X & Y \\ Y & X & 1 & Z \\ X & Y & Z & 1 \end{pmatrix}$$

which, according to (9), has only two independent parameters.

In terms of the angles, the determinant of the normalized metric gives the result[13]

$$\det(\hat{g}_{AB}) = -(2\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2.$$

The normalization (10) can be written by means of the *normalization matrix* $\left(M_A{}^{\hat{B}}\right)$,

$$\left(M_A{}^{\hat{B}}\right) = \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{pmatrix}$$

and its inverse, $\left(N_{\hat{A}}{}^B\right) = \left(M_A{}^{\hat{B}}\right)^{-1}$, expressing the basis transformation as

$$\hat{s}_{\hat{A}} = N_{\hat{A}}{}^B s_B, \qquad s_A = M_A{}^{\hat{B}}\hat{s}_{\hat{B}}.$$

The dual of this normalized basis, $\{\hat{s}_A\}$, gives a kind of *normalized* cobasis, $\{\hat{\ell}^A\}$, where

$$\hat{\ell}^{\hat{A}} \equiv M_B{}^{\hat{A}}\ell^B.$$

The covectors ℓ^A are null, thus it has no meaning to normalize them individually. However, this construction gives a criterion to normalize a set of four null covectors forming an emission tetrad. The normalized metric of this normalized cobasis, $\hat{\ell}^A\cdot\hat{\ell}^B \equiv \hat{g}^{AB}$, is

$$(\hat{g}^{AB}) = (\hat{g}_{AB})^{-1} = \frac{1}{2abc}\begin{pmatrix} 0 & c & b & a \\ c & 0 & a & b \\ b & a & 0 & c \\ a & b & c & 0 \end{pmatrix}, \tag{11}$$

where

$$a = \sin\theta_{23}, \qquad b = \sin\theta_{13}, \qquad c = \sin\theta_{12}.$$

Observe that while in the covariant metric there appear the cosine of the angles (8), in the contravariant metric there appear the sines.

The conditions for a, b, c can be proved to be equivalent to the equation

$$a^4 + b^4 + c^4 - 2a^2b^2 - 2a^2c^2 - 2b^2c^2 + 4a^2b^2c^2 = 0$$

and the inequalities $a, b, c > 0$. By the appropriate change we can denote

$$(\hat{g}^{AB}) = \begin{pmatrix} 0 & \hat{C} & \hat{B} & \hat{A} \\ \hat{C} & 0 & \hat{A} & \hat{B} \\ \hat{B} & \hat{A} & 0 & \hat{C} \\ \hat{A} & \hat{B} & \hat{C} & 0 \end{pmatrix}. \tag{12}$$

Then, the condition for $\hat{g}^{AB}$ to be the normalized metric of a normalized emission frame is[14]

$$\hat{A}^4 + \hat{B}^4 + \hat{C}^4 - 2\hat{A}^2\hat{B}^2 - 2\hat{A}^2\hat{C}^2 - 2\hat{B}^2\hat{C}^2 + 2\hat{A}\hat{B}\hat{C} = 0, \tag{13}$$

with $\hat{A}, \hat{B}, \hat{C} > 0$.

[13] The determinant of the original (non normalized) metric is

$$\det(g_{AB}) = -(2\mu_1\mu_2\mu_3\mu_4\sin\theta_{12}\sin\theta_{13}\sin\theta_{23})^2,$$

which gives a completely factorized expression.

[14] This equation can be alternatively computed from the determinant

$$\det(\hat{g}^{AB}) = \hat{A}^4 + \hat{B}^4 + \hat{C}^4 - 2\hat{A}^2\hat{B}^2 - 2\hat{A}^2\hat{C}^2 - 2\hat{B}^2\hat{C}^2 = \frac{-1}{(2abc)^2} = -2\hat{A}\hat{B}\hat{C}.$$

5.2. The splitting of the metric

We have arrived at the normalized emission cobasis $\{\hat{\ell}^A\}$ by the normalization of the vectors in the reciprocal basis, $\{\hat{s}_A\}$. This normalization is clearly related to the following splitting of the covariant and contravariant metrics:

$$g_{AB} = M_A{}^{\hat{C}} \hat{g}_{\hat{C}\hat{D}} M_B{}^{\hat{D}} \qquad \text{and} \qquad g^{AB} = N_{\hat{C}}{}^A \hat{g}^{\hat{C}\hat{D}} N_{\hat{D}}{}^B.$$

Or, written in matrix notation:

$$(g_{AB}) = -\begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{pmatrix} \begin{pmatrix} 1 & Z & Y & X \\ Z & 1 & X & Y \\ Y & X & 1 & Z \\ X & Y & Z & 1 \end{pmatrix} \begin{pmatrix} \mu_1 & 0 & 0 & 0 \\ 0 & \mu_2 & 0 & 0 \\ 0 & 0 & \mu_3 & 0 \\ 0 & 0 & 0 & \mu_4 \end{pmatrix}^{\mathsf{T}}$$

and

$$(g^{AB}) = \begin{pmatrix} \mu^1 & 0 & 0 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & \mu^3 & 0 \\ 0 & 0 & 0 & \mu^4 \end{pmatrix} \begin{pmatrix} 0 & \hat{C} & \hat{B} & \hat{A} \\ \hat{C} & 0 & \hat{A} & \hat{B} \\ \hat{B} & \hat{A} & 0 & \hat{C} \\ \hat{A} & \hat{B} & \hat{C} & 0 \end{pmatrix} \begin{pmatrix} \mu^1 & 0 & 0 & 0 \\ 0 & \mu^2 & 0 & 0 \\ 0 & 0 & \mu^3 & 0 \\ 0 & 0 & 0 & \mu^4 \end{pmatrix}^{\mathsf{T}}, \qquad (14)$$

where $\mu^A \equiv 1/\mu_A$.

The interesting point of this splitting is that the metric appears to be expressed in terms of two clearly different types of parameters: four parameters of 'scale' $(\mu_1, \mu_2, \mu_3, \mu_4)$ and two parameters of 'shape' (X, Y, Z with the constraint (9) or $\hat{A}, \hat{B}, \hat{C}$ with the constraint (13)). They have the following properties.

- The shape parameters are invariant with respect to the rescaling of the covectors $\ell^A = \mathrm{d}\tau^A$, or of the coordinates:

  $$\tau^A \mapsto \tau'^A = f^A(\tau^A).$$

  Therefore, $\hat{A}, \hat{B}, \hat{C}$ are independent of the particular time scales broadcasted by the emitters; they only depend on the trajectories (worldlines) of the emitters.

  Furthermore, they are invariant with respect to conformal transformations of the metric, depending only on the causal structure of the spacetime.
- Each of the scale parameters μ^A is invariant with respect to the rescaling of the rest of covectors $\mathrm{d}\tau^B$, $B \neq A$. Thus, it is invariant with respect to the particular time scales broadcasted by the rest of emitters, depending only on the clock of the corresponding emitter. Moreover, it is proportional to the rescaling of the corresponding covector $\mathrm{d}\tau^A$. This means that, for the change of coordinates representing the deviation of one of the clocks,

  $$\tau^A \mapsto f(\tau^A) \qquad \text{and} \qquad \tau^B \mapsto \tau^B \quad \forall B \neq A,$$

  we have

  $$\mu^A \mapsto \dot{f}\mu^A \qquad \text{and} \qquad \mu^B \mapsto \mu^B \quad \forall B \neq A,$$

  where $\dot{f}$ is the derivative of f.

Nevertheless, let us note here that, with respect to the change in the trajectory of the emitters, the modification of just one of them will change all the parameters in a non-trivial way.

5.3. The splitting of the metric in three dimensions

We could have introduced the same kind of splitting presented here for the four-dimensional metrics also for the tridimensional ones. The deduction is analogous, but with some simplifications. For instance, there are only three angles $\theta_{12}, \theta_{13}, \theta_{23}$, and they all coincide: $\theta_{AB} = \pi$. The resulting splitting is thus much more simple:

$$\begin{aligned}(g_{AB}) &= \begin{pmatrix}\mu_1 & 0 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & \mu_3\end{pmatrix}\begin{pmatrix}-1 & 1 & 1\\ 1 & -1 & 1\\ 1 & 1 & -1\end{pmatrix}\begin{pmatrix}\mu_1 & 0 & 0\\ 0 & \mu_2 & 0\\ 0 & 0 & \mu_3\end{pmatrix}^{\mathsf{T}}\\ (g^{AB}) &= \frac{1}{2}\begin{pmatrix}\mu^1 & 0 & 0\\ 0 & \mu^2 & 0\\ 0 & 0 & \mu^3\end{pmatrix}\begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}\mu^1 & 0 & 0\\ 0 & \mu^2 & 0\\ 0 & 0 & \mu^3\end{pmatrix}^{\mathsf{T}}.\end{aligned} \tag{15}$$

Observe that there are three scale parameters, given by

$$\mu^A = \sqrt{\frac{2g^{AB}g^{AC}}{g^{BC}}}, \qquad \text{with} \quad A, B, C \neq,$$

and no shape parameter. Accordingly, the normalized metrics, $\hat{g}_{AB}$ or $\hat{g}^{AB}$, have no degrees of freedom.

6. The covariant metric from the contravariant metric

At this point one is led to relate the parameters for the metric obtained in the preceding section with the components of the contravariant metric given as the starting point in formula (1) of section 3. The best method to obtain this relationship is computing the inverse of the contravariant metric.

6.1. The inverse of the contravariant metric

The covariant metric, g_{AB}, is obtained from the contravariant metric, g^{AB}, by inverting the corresponding matrix. Recalling that the determinant of the contravariant metric is given by the Heron polynomial (5), $\det(g^{AB}) = -\mathcal{H}(\mathcal{A}, \mathcal{B}, \mathcal{C})$, of the parameters A, B, C defined by formula (3), the resulting covariant metric is

$$(g_{AB}) = \frac{1}{\mathcal{H}(A, B, C)}\begin{pmatrix}2g^{23}g^{24}g^{34} & \gamma g^{34} & \beta g^{24} & \alpha g^{23}\\ \gamma g^{34} & 2g^{13}g^{14}g^{34} & \alpha g^{14} & \beta g^{13}\\ \beta g^{24} & \alpha g^{14} & 2g^{12}g^{14}g^{24} & \gamma g^{12}\\ \alpha g^{23} & \beta g^{13} & \gamma g^{12} & 2g^{12}g^{13}g^{23}\end{pmatrix},$$

where $\alpha \equiv A^2 - B^2 - C^2$, $\beta \equiv B^2 - A^2 - C^2$, $\gamma \equiv C^2 - A^2 - B^2$. The components can be compactly written as

$$\begin{aligned}g_{AA} &= \frac{-2}{\mathcal{H}}g^{BC}g^{BD}g^{CD}\\ g_{AB} &= \frac{1}{\mathcal{H}}g^{CD}(g^{AB}g^{CD} - g^{AC}g^{BD} - g^{AD}g^{BC})\end{aligned} \tag{16}$$

for $A, B, C, D \neq$. Observe that this gives the relationship

$$g_{AB}/g^{CD} = g_{CD}/g^{AB}. \tag{17}$$

6.2. The algebraic confirmation of the geometric properties

The components of the covariant metric gives the scalar products between the natural basis of vectors of emission coordinates, $g_{AB} = s_A{\cdot}s_B$, where $\{s_A \equiv \partial_A\}$ is the reciprocal basis of $\{\ell^A \equiv \mathrm{d}\tau^A\}$. Thus, we can observe in (16) that the four vectors are, as expected, spacelike:

$$s_A{\cdot}s_A = g_{AA} < 0 \qquad \forall A = 1, 2, 3, 4.$$

Thus, the scale parameters in terms of the components of the contravariant metric are given by

$$\mu_A = \sqrt{-g_{AA}} = \sqrt{\frac{2}{\mathcal{H}} g^{BC} g^{BD} g^{CD}}, \tag{18}$$

with $A, B, C, D \neq$.

We can also obtain the cosine of the angle between each pair of vectors:

$$\cos\theta_{AB} \equiv \frac{-g_{AB}}{\sqrt{-g_{AA}}\sqrt{-g_{BB}}},$$

which from equation (17) satisfy

$$\cos\theta_{AB} = \frac{-\mathcal{H} g_{AB}}{2g^{CD}\sqrt{g^{BC} g^{BD} g^{AC} g^{AD}}} = \cos\theta_{CD}$$

confirming the equality of the angles between complementary pairs stated in theorem 2. This gives us the result

$$\begin{aligned} X &\equiv \cos\theta_{23} = \cos\theta_{14} = \frac{A^2 - B^2 - C^2}{2BC} \\ Y &\equiv \cos\theta_{13} = \cos\theta_{24} = \frac{B^2 - A^2 - C^2}{2AC} \\ Z &\equiv \cos\theta_{12} = \cos\theta_{34} = \frac{C^2 - A^2 - B^2}{2AB}, \end{aligned} \tag{19}$$

which relates the scale invariant parameters X, Y, Z of the preceding section with the components of the contravariant metric.

Observe that (19) is almost the law of cosines for the triangle of sides A, B, C, differing only by having the opposite sign. This means that the angles θ_{AB} are the supplementary of the angles of this triangle:

$$\theta_{12} = \pi - \angle AB, \qquad \theta_{13} = \pi - \angle AC, \qquad \theta_{23} = \pi - \angle BC.$$

This geometrical relationship is illustrated in figure 3.

Then, from the well-known property for the sum of the angles of a triangle, $\angle AB + \angle AC + \angle BC = \pi$, it follows

$$\theta_{12} + \theta_{13} + \theta_{23} = 2\pi,$$

confirming the result of theorem 2. This can also be confirmed by checking that (19) implies the conditions (9).

6.3. The scale and shape parameters in terms of the metric components

In equations (18) and (19) we have already found the scale parameters μ_A and the shape parameters X, Y, Z in terms of the contravariant metric components. But we are also interested in obtaining the expression for the shape parameters $\hat{A}, \hat{B}, \hat{C}$. Let us recall that, from (11) and (12), they are given in terms of the sines of the angles by

$$\hat{A} = \frac{1}{2bc}, \qquad \hat{B} = \frac{1}{2ac}, \qquad \hat{C} = \frac{1}{2ab},$$

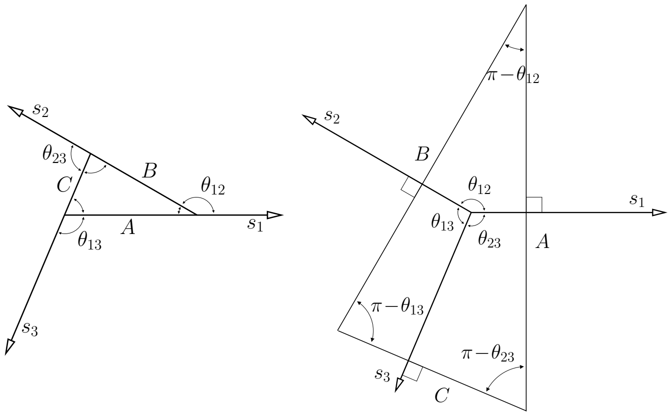

Figure 3. Geometrical relationship between the angles formed by the natural vectors, s_A, and the triangle of parameters, A, B, C. Two different visualizations.

where $a = \sin\theta_{23}, b = \sin\theta_{13}, c = \sin\theta_{12}$. Thus by using the identity $\sin\theta_{AB} = \sqrt{1-(\cos\theta_{AB})^2}$ in (19) we get

$$a = \frac{\sqrt{4B^2C^2-(A^2-B^2-C^2)^2}}{2BC} = \frac{\sqrt{\mathcal{H}}}{2BC}, \qquad b = \frac{\sqrt{\mathcal{H}}}{2AC}, \qquad c = \frac{\sqrt{\mathcal{H}}}{2AB}.$$

Therefore, we obtain the *normalization factor* for the parameters:

$$\hat{A} = \frac{2ABC}{\mathcal{H}}A, \qquad \hat{B} = \frac{2ABC}{\mathcal{H}}B, \qquad \hat{C} = \frac{2ABC}{\mathcal{H}}C. \tag{20}$$

From this expression and equation (18) we can check for instance that $g^{23} = \mu^2\mu^3\hat{A}$, which confirms the splitting of the contravariant metric given by (14).

6.4. Confirming the scale properties of the parameters

We have obtained the expressions for the shape parameters, $\hat{A}$, $\hat{B}$, $\hat{C}$, and the scale parameters, μ_A, in terms of the components of the contravariant metric. Now we check that they satisfy the expected properties when some of the emitted time scales is rescaled.

Let us consider, for instance, that the first emitter does not emit the expected time, τ^1, but emits a different time, related to the supposed one by

$$\tau^{1'} = f(\tau^1).$$

Then, the family of null cones emitted by this emitter will be relabelled and the natural covector will transform accordingly as

$$\mathrm{d}\tau^{1'} = \dot{f}\mathrm{d}\tau^1,$$

and the contravariant metric as

$$g^{1'A} = \dot{f}g^{1A},$$

with g^{AB} unchanged for $A, B \neq 1$.

Taking the expression (20) of the shape parameters, for instance,

$$\hat{A} = \frac{2ABC}{\mathcal{H}} A,$$

we can check its invariance:

$$\mathcal{H}' = \dot{f}^2 \mathcal{H}, \qquad A' = \sqrt{\dot{f}} A, \qquad B' = \sqrt{\dot{f}} B, \qquad C' = \sqrt{\dot{f}} C \qquad \Rightarrow \qquad \hat{A}' = \hat{A}.$$

And taking the expression (18) of the scale parameters:

$$\mu^1 = \sqrt{\frac{\mathcal{H}}{2g^{23}g^{24}g^{34}}} \qquad \text{and} \qquad \mu^2 = \sqrt{\frac{\mathcal{H}}{2g^{13}g^{14}g^{34}}},$$

we can check that

$$\mu^{1'} = \dot{f} \mu^1 \qquad \text{and} \qquad \mu^{2'} = \mu^2.$$

7. Conclusion

In this paper we have studied the causal and geometric nature of the natural basis and cobasis of emission coordinates in four-dimensional spacetimes, generated by four emitters broadcasting proper time scales. We have obtained some interesting non-trivial properties that characterize this class of coordinates. The most remarkable result is obtaining a natural splitting of the metric involving the separation of its six degrees of freedom into two different types of parameters, which behave in clearly different ways when the particular time scales broadcasted by the emitters are changed: two parameters independent of the four time scales and four parameters, one for each emitter, dependent only on the corresponding scales. This parametrization should simplify the construction of the positioning systems, in decoupling those characteristics depending on the trajectories of the emitters from those that depend on the time broadcasted.

The relativistic positioning systems based in emission coordinates are quite well developed for two-dimensional spacetimes. However, the known results for the two-dimensional case are not trivially generalizable for higher dimensions. This is the first quantitative paper aimed to develop an exact theory of relativistic positioning systems in four-dimensional spacetimes. In a following paper we will present properties relating the perception of emission coordinates by an arbitrary observer and the existence of special observers [28]. Also we will present some global properties of these coordinates implied by the condition of causality in the spacetime [27], and will study the positioning system obtained for some simple particular cases in Minkowski spacetimes [26].

Appendix. General algorithm for the mathematical construction of emission coordinates

We present here a procedure to evaluate the emission coordinates generated by four known emitters in any given spacetime. A general method is sketched in appendix A.1 and its set in equations is presented in appendix A.2. In appendix A.3 we mention two methods to find the field of light cones and in appendix A.4 we present a two-dimensional example.

A.1. General description.

Emission coordinates being broadcasted by light signals between finitely separated events, their construction in a mathematical model of a physical situation needs the knowledge of the light cones of the spacetime in question, in the region of interest, with the precision required by the physical situation.

Suppose these cones known (see appendix A.3 below) and, for any event x of this region, let us denote respectively the future and past light cones of vertex x by $\mathcal{C}_x^+$ and $\mathcal{C}_x^-$.

Also, let $\gamma_A(\tau^A)$ (no summation) be the trajectories of the emitters (not necessarily geodesic!) already parametrized with proper time scales τ^A.

Then, the four families of τ^A-parametrized coordinate hypersurfaces are given by $\mathcal{C}^+_{\gamma_A(\tau^A)}$. So, the emission coordinates (τ^A) of an event y are constituted by the four values τ^A for which one has $y \in \mathcal{C}^+_{\gamma_A(\tau^A)}$, or equivalently

$$y \cup \mathcal{C}^+_{\gamma_A(\tau^A)} = \mathcal{C}^+_{\gamma_A(\tau^A)}. \tag{A.1}$$

Observe that, because the past null cone contains every incoming null geodesic, in particular the four geodesics followed by the signals reaching the event from the four emitters, (A.1) may alternatively be written:

$$\mathcal{C}_y^- \cap \gamma_A(\tau^A) = \gamma_A(\tau^A). \tag{A.2}$$

Depending on the form under which is known the information on the light cone, this last equation may be much easier to handle than (A.1). Emission coordinates $\{\tau^A\}$ follow for generic y in the region. From them, the expression of the metric in emission coordinates is obtained from its expression in the initial coordinates in the standard way.

A.2. Local equations.

In practice, the simplest local form that the light cones can adopt is that of a function of two points x and y such that it vanishes if and only if y belongs to the corresponding cone of vertex x, or conversely. For example, $\mathcal{C}_x^+ \equiv \kappa^+(x, y)$ such that the event y is in $\mathcal{C}_x^+$ if and only if $\kappa^+(x, y) = 0$.

Then, the local equation of the four families of τ^A-parametrized coordinate hypersurfaces $\mathcal{C}^+_{\gamma_A(\tau^A)}$ is $\kappa^+(\gamma_A(\tau^A), y) = 0$.

Corresponding to equation (A.1), the coordinates (τ^A) of an event y are the solutions in τ^A of the four equations

$$\kappa^+(\gamma_A(\tau^A), y) = 0 \tag{A.3}$$

for the given y. For a generic point y, of initial arbitrary coordinates $\{y^\alpha\}$, these equations may be solved alternatively in τ^A or in y^α, given the corresponding coordinate transformations to emission coordinates: $\tau^A = \Theta^A(y^\alpha)$ or its inverse $y^\alpha = \mathcal{Y}^\alpha(\tau^A)$.

Observe that if generically one has $\mathcal{C}_x^+ \equiv \kappa^+(x, y)$ for any x, then the past light cones $\mathcal{C}_x^- \equiv \kappa^-(x, y)$ are such that $\kappa^-(x, y) = \kappa^+(y, x)$. In this case one clearly sees that the local expression of equation (A.2),

$$\kappa^-(y, \gamma_A(\tau^A)) = 0$$

is strictly equivalent to equation (A.3).

A.3. On the light cones.

The ways to obtain the light cones depend on the starting information that one has on the spacetime in question. In principle, they may be obtained by integration of the equations of null geodesics. Nevertheless, because this implies more information than the necessary one for the task (the light cone is only the *enveloping* of classes of null geodesics), in general this procedure may involve, and frequently involves, many unnecessary difficulties. A more simple (less complicated) way may be that of obtaining the *world function* $\Omega(x, y)$ [29] (essentially: one-half of the square of the geodesic distance between x and y). Hamilton–Jacobi equations are certainly useful in post-Newtonian situations (see, for example [30]), although a direct method is the integration of its *characteristic equations*:

$$g(x)^{\alpha\beta}\frac{\partial}{\partial x^\alpha}\Omega(x, y)\frac{\partial}{\partial x^\beta}\Omega(x, y) = 2\Omega(x, y)$$
$$g(y)^{\alpha\beta}\frac{\partial}{\partial y^\alpha}\Omega(x, y)\frac{\partial}{\partial y^\beta}\Omega(x, y) = 2\Omega(x, y).$$

The solutions in y of $\Omega(x, y) = 0$ are either in $\mathcal{C}_x^+$ or in $\mathcal{C}_x^-$.

A.4. A two-dimensional example.

Let us suppose that, of an arbitrary two-dimensional Lorentzian metric g we have been able to find one conformally flat scalar factor m and that, of the corresponding flat metric, we have been able to find its Cartesian or inertial coordinates $\{t, x\}$. This means that we are able to write the metric g in the form

$$g(t, x) = m(t, x)\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}.$$

In two dimensions, as every one knows, the light cones consist of only two null geodesics, that we shall distinguish here as the *right directed* $\mathcal{R}$ and the *left directed* $\mathcal{L}$. So, the light cone of vertex $(\bar{t}, \bar{x})$ is given by the two null geodesics:

$$\mathcal{R} \equiv \{(t - \bar{t}) - (x - \bar{x}) = 0\}, \qquad \mathcal{L} \equiv \{(t - \bar{t}) + (x - \bar{x}) = 0\}\}. \tag{A.4}$$

On the other hand, of a timelike trajectory γ of local equation $\gamma \equiv \{t = t(\tau); x = x(\tau)\}$, τ is a proper time scale relative to the metric g if the functions $t(\tau)$ and $x(\tau)$ verify

$$m(t(\tau), x(\tau))\{\dot{t}^2(\tau) - \dot{x}^2(\tau)\} = 1.$$

Consider now two emitters of respective trajectories $\gamma_1 \equiv \{t = t_1(\tau^1); x = x_1(\tau^1)\}$ and $\gamma_2 \equiv \{t = t_2(\tau^2); x = x_2(\tau^2)\}$, and suppose that one of them, say γ_1, always remains to the left of the other in the plane $\{t, x\}$. It is easy to understand that emission coordinates exists only in the region between both trajectories and that in this region the light cones with vertex on γ_1 (resp. γ_2) are defined by the sole null geodesics $\mathcal{R}_1$ (resp. $\mathcal{L}_2$). Taking into account (A.4), the coordinate 'hypersurfaces' (here 'lines') of emission coordinates are then given by

$$\begin{aligned}(t - t_1(\tau^1)) - (x - x_1(\tau^1)) = 0\\ (t - t_2(\tau^2)) + (x - x_2(\tau^2)) = 0.\end{aligned} \tag{A.5}$$

It directly follows that the coordinate transformation *from* emission coordinates $\{\tau^1, \tau^2\}$ to conformal inertial ones $\{t, x\}$ is given by

$$\begin{aligned}t = \tfrac{1}{2}[t_1(\tau^1) - x_1(\tau^1) + t_2(\tau^2) + x_2(\tau^2)]\\ x = \tfrac{1}{2}[-t_1(\tau^1) + x_1(\tau^1) + t_2(\tau^2) + x_2(\tau^2)].\end{aligned} \tag{A.6}$$

The inverse transformation, giving the emission coordinates in terms of the conformal inertial ones, needs the knowledge of the explicit form of the equations $\{t = t_A(\tau^A), x = x_A(\tau^A)\}$. Such equations have been obtained elsewhere for some cases of physical interest (see [2, 25]).

In more than two dimensions, the calculations are less easy. Nevertheless, the four-dimensional analogues to equations (A.6) in all the generality are under writing. It is to be noted that, although the starting idea of emission coordinates is simple, our lack of intuition at present about four light-like coordinates is naturally accompanied by many prejudices, which have to be slowly and carefully eliminated if we want a good comprehension of these emission coordinates. For example, emission coordinates are homogeneous in the sense that every one of them is constructed with similar devices and broadcasts signals in similar way (in contrast with the usual ones, more or less directly associated with three rods and one clock). Who, then, would guess that such homogeneous physical coordinates, restrict the space $\{\tau^A\}$ to an (n-1) dimensionally compacted cylindrical region, or guess that in this physical region there are subregions double-valuated by the physical signals, as shown in [26]? The exploration of emission coordinates offers a vast field of which we are at present unaware of almost all its features.

Acknowledgments

J M Pozo acknowledges the support of the postdoc fellowship EX-2004-0090, from the spanish *Ministerio de Educación y Ciencia*, and the project BFM2003-07076.

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