PHYSICAL RELATIVISTIC FRAMES

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ABSTRACT. The concept of relativistic positioning system is introduced and the project SYPOR, aiming for its application to the Galileo navigation system, is sketched.

1. INTRODUCTION

Coordinate systems are the basic pieces for the construction of reference systems and, consequently, of reference frames. There is an abundant literature on them, but they are always considered from partial points of view. This is true both, in newtonian physics as well as in relativity. Moreover, an extended idea pretends that there is no essential differences between newtonian and relativistic coordinate systems.

It is clear that further thought, based in larger points of view, needs to be given to this subject. To provide incentives in this direction, I will begin here with a fairly basic presentation of the different classes of coordinate systems admitted by both, newtonian and relativistic space-times. This is the subject of Section 2.

Able to coincide in newtonian physics, reference systems and positioning systems are necesarily disjoint objects in relativity. The notion of relativistic positioning systems, their basic properties and their simple possibilities of construction are presented in Section 3.

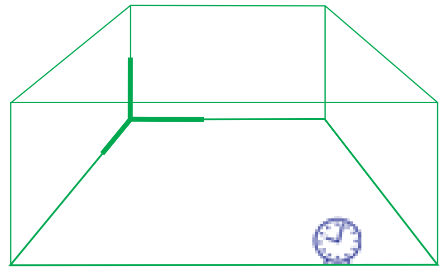
The idea of the project SYPOR appear then clearly: to give to the constellation of satellites of the future GALILEO navigation system the status of a *primary relativistic positioning system for the Earth*. Points concerning this idea, and a simple version of an important control result are briefly sketched in Section 4.

Some ideas and results on this subject come from a long collaboration with Ll. Bel, J.J. Ferrando, J.A. Morales and A. Tarantola.

2. CAUSAL CLASSES OF COORDINATE SYSTEMS

In newtonian space-time, one uses to take coordinate systems in which one coordinate line is generated by a clock and the other three, more or less directly related to rods, lie in the absolute three-dimensional space at every instant. The space-time line generated by the clock, because transverse to this space at any instant, is called *time-like*, and those determined by the rods, because belonging to it, are called *space-like*; the corresponding coordinate systems are said of *type* **teee**.

Figure 1: A coordinate system is usually related to a clock and three rods.



But coordinate systems other than the above ones may also be taken. For example, those constituted by four separated megaphones shouting at every instant the time of a clock in a region of the space (the gap at every point between their messages, due to the sound's velocity, allow to locate the point in the space-time). It can be seen that now the corresponding four coordinate lines are *time-like* lines; these coordinate systems are said of *type* **tttt**.

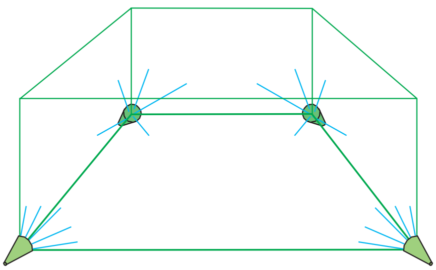


Figure 2: Coordinate systems may also be realized by means of four megaphones shouting at every instant the time of a clock.

Coordinate systems of types **teee** and **tttt** are said to belong to different *causal classes*[1]. Thus, a simple and natural question arises: *how many causal classes of coordinate systems exist in Newtonian space-time?*

The fact that this natural question has never been asked shows the above mentioned partial character of the studies made up to now on coordinate systems, reference systems and reference frames. The answer is here presented to information only, as a mean to emphasize the differences between Newtonian and relativistic coordinate systems.

This answer says that *there exist twelve disjoint causal classes of Newtonian coordinate systems*, according to Table I. In this table, t's and e's denote respectively time-like and space-like causal characters, and italic, roman and capital styles denote respectively coordinate hypersurfaces, coordinate lines and coordinate surfaces (or covectors, tangent vectors and planes rspectively). A class is represented by the set $\{\boldsymbol{xxxx}; \mathbf{XXXXXX}; \mathbf{xxxx}\}$ of the causal characters of the coordinate ingredients of a coordinate system. The six surfaces or planes of every class $\mathbf{X}_1\mathbf{X}_2\mathbf{X}_3\mathbf{X}_4\mathbf{X}_5\mathbf{X}_6$ are subtended respectively by the four hypersurfaces $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, $\boldsymbol{x}_3$, $\boldsymbol{x}_4$ in the following order: $\mathbf{X}_1 = \boldsymbol{x}_1 \wedge \boldsymbol{x}_2$, $\mathbf{X}_2 = \boldsymbol{x}_1 \wedge \boldsymbol{x}_3$, $\mathbf{X}_3 = \boldsymbol{x}_1 \wedge \boldsymbol{x}_4$, $\mathbf{X}_4 = \boldsymbol{x}_2 \wedge \boldsymbol{x}_3$, $\mathbf{X}_5 = \boldsymbol{x}_2 \wedge \boldsymbol{x}_4$, $\mathbf{X}_6 = \boldsymbol{x}_3 \wedge \boldsymbol{x}_4$.

What causal classes correspond in relativity to these twelve classes? The absolute lack of intuition about the answer evidence the need of further thought on this subject: *in relativity there exist one hundred and ninety nine causal classes of coordinate systems* (Coll and Morales, 1992, Coll 2000). They are given in Table II, where now $\boldsymbol{l}$, $\mathbf{l}$ and $\mathbf{L}$ stand respectively for light-like coordinate hypersurfaces, lines and surfaces.

On these one hundred and ninety nine different ways of locating points of our neighbourhood, only *two* have been significantly considered in the literature: the "classical" one, $\{\boldsymbol{teee}; \mathbf{TTTEEE}; \mathbf{teee}\}$, and, for theoretical considerations, the "null" ones, $\{\boldsymbol{llee}; \mathbf{TLLLLE}; \mathbf{llee}\}$, associated to some radiative problems. The reference systems belonging to all the other *nine hundred and ninety seven* classes remain essentially unexplored[2] .

[1] For the notion of causal class in relativity, see (Coll and Morales, 1992).

[2] Reference systems with light-like coordinate lines were considered in Coll, 1985; for their causal duals, i.e.

	teee	ttee	ttte	tttt
eeee	TEEEEE EEEEEE	TEEEEE EEEEEE	TEEEEE EEEEEE	TEEEEE EEEEEE
teee	TTTEEE	TTTEEE	TTTEEE	TTTEEE

TABLE I: The twelve causal classes of reference systems of Newtonian physics.

Perhaps the origin of this sort of cultural alienation is the fact that our vision of the world is, due to many historical reasons, too polarized in the evolution point of view of the space-time. A plausible vision, of course, but a very particular one in the set of the almost two hundred other points of view... Our group *Systèmes de Référence Relativistes* of the DANOF is particularly concerned by the study of coordinate systems with potential interest in astronomy and physics, watever be their character, usual or unusual.

3. POSITIONING SYSTEMS

One of the reasons that make coordinate systems important in practice is the fact that they allow *to situate points of a region with respect to one observer* (usually located at the origin); this function gives rise to *reference systems* (resp. *reference frames*). But another important reason for their use is that they allow *to indicate to every point its position* (with respect to them); this gives rise to *positioning systems* (resp. *positioning frames*).

In Newtonian theory, coordinate systems may be constructed in such a way that both functions be simustaneously accomplished. In relativity, nevertheless, this is not possible by a sole coordinate system. Thus, in relativity we are led to consider separately both, reference systems (resp. reference frames) and positioning systems (resp. positioning frames). We are here interested in positioning systems.

In relativity, positioning systems are intended to have three important properties, namely those of being *generic*, *free* and *immediate*.

* A relativistic positioning system is *generic* for a given class of space-times if the underlying coordinate system exists in *any* space-time of this class. For example, cartesian systems are not generic but for (the class of) the sole flat space-time; harmonic systems are generic for the whole class of all space-times.

* It is *free*, or *gravity-free*, if their (physical) construction does not need the knowledge (measure) of the gravitational field (space-time metric). For example, harmonic systems are not free.

* It is *immediate* if every point of its space-time domain may know its coordinates without delay, in real time. For example, the inertial system constructed by an inertial observer by means of two-way signals is not immediate.

An important epistemic result is that the set of generic, free and immediate relativistic positioning systems constitute a little class of systems. The simplest one of the class consists of four free falling clocks (satellites) broadcasting their proper times.

The coordinate systems associated to these positioning systems are thus such that their coordinate hypersurfaces are constituted by the loci of equal proper time of every satellite.

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with light-like coordinate hypersurfaces, see Coll 2000 and also Hehl 2001.

	eeee	leee	elee	teee	llee	tlee	ttee	llle	tlle	ttle	ttte	llll	tlll	ttll	tttl	tttt
eeee	EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE LLLLEE TLLLEE TTLLEE TTTLEE TTTTEE LLLLLE TLLLLE TTLLLE TTTLLE TTTTLE TTTTTE LLLLLL TLLLLL TTLLLL TTTLLL TTTTLL TTTTTL TTTTTT	EEEEEE LEEEEE EEELEE TEEEEE LELEEE LEELEE EELLEE TLEEEE TEELEE EETLEE TTEEEE LLLEEE LLELEE LELLEE TLLEEE TLELEE TELLEE LETLEE TTLEEE TTELEE TETLEE TTTEEE LLLLEE TLLLEE LLTLEE TTLLEE TLTLEE TTTLEE		EEEEEE LEEEEE TEEEEE LLEEEE TLEEEE TTEEEE LLLEEE TLLEEE TTLEEE TTTEEE	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE EELLEE ELELEE TLEEEE LLELEE LELLEE TLELEE TELLEE	EEEEEE LEEEEE ELEEEE TEEEEE LLEEEE TLEEEE	EEEEEE LEEEEE TEEEEE	EEEEEE LEEEEE LLEEEE LLELEE	EEEEEE LEEEEE LLEEEE	EEEEEE LEEEEE	EEEEEE	EEEEEE	EEEEEE	EEEEEE	EEEEEE	EEEEEE
leee	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE TTLTEE TTLETL TTTTEE TTLLLE TTLELL TTTLLE TTLTLE TTLLTE TTLETL TTTTLE TTLTTE TTLETT TTTTTE TTLLLL TTTLLL TTLTLL TTLLTL TTTTLL TTLTTL TTLLTT TTTTTL TTLTTT TTTTTT	TTLEEE TTTEEE TTLLEE TTLELE TTTLEE	TLLEEE TLLLEE TLLTEE TLLLLE TLLTLE TLLTTE	TTLEEE TTTEEE	TLLEEE TLLLEE	TLLEEE										
teee	TTTEEE TTTLEE TTTTEL TTTLLE TTTTLE TTTTTE TTTLLL TTTTLL TTTTTL TTTTTT	TTTEEE TTTLEE		TTTEEE												
llee	TTLTLE TTLLTE TTTTLE TTTTTE TTLTLL TTLLTL TTTTLL TTLLTT TTLTLT TTTTTL TTTTLT TTTTTT	TTLLLE TTTLLE			TLLLLE											
tlee	TTTTLE TTTTTE TTTTLL TTTTTL TTTTLT TTTTTT	TTTLLE														
ttee	TTTTTE TTTTTL TTTTTT															
llle	TTLTLL TTTTLL TTTTTL TTTTTT															
tlle	TTTTLL TTTTTL TTTTTT															
ttle	TTTTTL TTTTTT															
ttte	TTTTTT															
llll	TTTTTT															
tlll	TTTTTT															
ttll	TTTTTT															
tttl	TTTTTT															
tttt	TTTTTT															

t , l , e — hypersurfaces or covectors
t , l , e — lines or tangent vectors
T , L , E — surfaces or planes

are respectively time-like , light-like , space-like coordinate

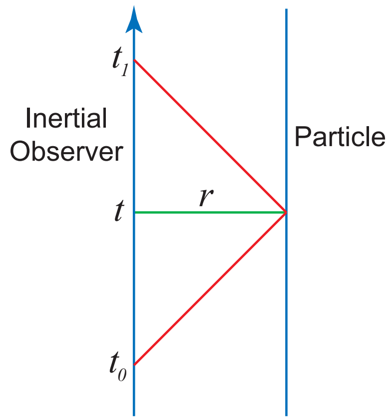
The 6 surfaces or planes of every class $X_1X_2X_3X_4X_5X_6$ are generated respectively by the 4 hypersurfaces or covectors x_1, x_2, x_3, x_4 in the following order

$X_1 = x_1 \wedge x_2$ $X_2 = x_1 \wedge x_3$ $X_3 = x_1 \wedge x_4$

$X_4 = x_2 \wedge x_3$ $X_5 = x_2 \wedge x_4$ $X_6 = x_3 \wedge x_4$

TABLE II: The 199 causal classes of reference systems of relativistic physics.

Figure 3: The location of points by an observer using two-way signals does not constitute an immediate system.



They are thus null hypersurfaces. According to Table II, all these coordinate systems belong to a sole causal class, namely that of the form $\{\boldsymbol{llll}; \mathbf{TTTTTT}; \mathbf{eeee}\}$, a very unusual one (for them, all coordinates, like the usual time coordinate, "flow" whatever the observer). A two dimensional representation would have the aspect shown in figure 4.

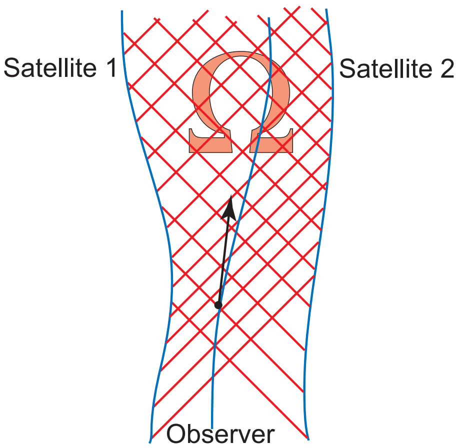


Figure 4: Two satellites broadcasting their proper time by means of electromagnetic signals constitute a relativistic positioning system

4. THE PROJECT *SYPOR*

All usual coordinate systems on the Earth are Newtonian. But constellations of satellites broadcasting their proper times are, of own right, relativistic physical systems. Rather than to flatten them directly against an approximate Newtonian model, it is better to treat conceptually them as principal objects.

The objective of the project SYPOR[3] is to endow the Earth with a relativistic positioning system. More precisely, it aims to use the constellation of satellites of the future GALILEO navigation system as an immediate, generic, free and primary relativistic positionning system for the Earth.

Every four neighbouring satellites of the constellation generate a *local chart*, the constellation defining then the *primary atlas* of local charts for the surrounding area of the Earth.

Newtonian cartographic or geocentric coordinates become in the project secondary coordinate systems, to be defined with respect to this primary atlas attached to the constellation. This defining task concerns the control segment of the navigation system, which must invert its usual reading, its function being not now to determine the position of the satellites with respect

[3]Presented at the CNES "appel à idées" of September 2001

to some terrestrial coordinates but to define these last ones with respect to the constellation of satellites.

In order to complete the primary character of the constellation, one has i) to endow every satellite with a device sending its proper time to the neighbouring satellites (autonomy of control and internal conformation), ii) to endow at least four satellites of a device for orientation with respect to the ICRF (autonomy of control and external orientation), and iii) to endow every satellite of a device broadcasting over the Earth its *proper coordinates*, i.e. not only its proper time, but the proper times received from the neighboring satellites (guarantee of *public character*, defining completely the system and allowing authorized users to evaluate the precision of the system at every instant).

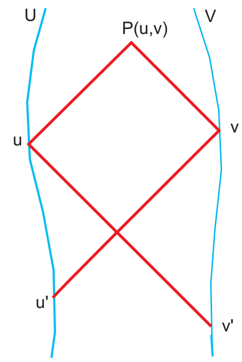


Figure 5: The location of points by an observer using two-way signals does not constitute an immediate system.

Finally, to close this survey, we shall present a two-dimensional simplified version of the complete definition of the system. In the coordinate system $\{u, v\}$ generated by the proper times of the satellites U and V, any observer receives; at every instant (u, v) of its trayectory, the coordinates (u, v') of U and the coordinates (v, u') of V. These sets of values up to the instant P define, in the coordinate system $\{u, v\}$, the trayectories $\Phi(u, v) = 0$ and $\Psi(u, v) = 0$ of the satellites U and V respectively. One can then show the following result:

Theorem (Coll-Morales-Tarantola): *In a two-dimensional space-time, let U and V be two arbitrary satellites emitting respectively their proper times u and v and their reciprocal ones u' and v'. Let $\Phi = 0$ and $\Psi = 0$ be respectively the equations of their trayectories, obtained from the sent data. Then, in the coordinate system $\{u, v\}$ so generated, the space-time metric tensor g is given by*

$$g = \frac{1}{2} \Phi'_u \, \Psi'_v \, du \otimes dv \ .$$

This information is all what one can expect to know from a primary system of coordinates, so it is complete. Observe that, in terms of these proper times u and v, the expression is independent not only of the velocities of the satellites in their constellation (relative velocities), but also of the velocity of the user with respect to the constellation.

5. REFERENCES

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