LIGHT COORDINATES IN RELATIVITY *

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Abstract

The construction of coordinate systems by intersection of four beams of light is analyzed.

1 INTRODUCTION

a) The *principle of local covariance* of Classical Physics[1] asserts that every observer can formulate the physical laws in a form independent of the coordinate systems. From an epistemic point of view, this principle may be considered as an extension of the *principle of dimensional invariance*, which asserts that physical laws can be formulated in a form independent of the system of units[2]. In Relativistic Physics, the *principle of general covariance* reinforce even more the validity of the above one of local covariance[3].

A first consequence of these principles is the mathematical representation of physical quantities by geometric objects related to the corresponding invariance groups. Thus, in the general formulation of laws, *where specific relations between physical quantities are stated*, coordinate systems actually play a superfluous role. Nevertheless, in the establishment, verification and

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utilization of every physical law, *where specific values of every physical quantity intervene*, the chosen coordinate systems (as well as the chosen units), have to be the object of a rigorous and precise operational description [4].

In Classical Physics, to the hypothesis of absolute time and absolute and flat space, it is added the assumption that there is no difficulty of principle for the construction of spatial lines, particularly geodesics. It result that the geometrically admissible coordinate systems, and particularly the cartesian ones, may always be described, more or less directly, in a operational form.

In Relativity the situation is, nevertheless, very different. On one hand, it is not possible to associate, to *all* observers, cartesian coordinate systems and, on the other hand, even for those that geometrically admit them, it is not easy to give, and it has been not done up to now, an *operational* definition of them[5]. The situation for non cartesian coordinate systems is even more obscure.

b) Among all curves, the easier to materialize in Relativity are, undoubtedly, light-like geodesics: for this task it is sufficient "to liberate photons". Hence the idea to use such curves as coordinate lines. More precisely, we intend here to start the *study of those space-time coordinate systems whose coordinate lines are constituted by four congruences of light-like geodesics*.

In the domains of the space-time where such systems exist, the coordinate lines may be materialize by four light beams[6], the admissible parameterizations on every line being defined by the frequencies of every beam.

The conceptual interest of these systems is not only their simple operational definition, but also their totally metric character: for every observer, the values of the four frequencies and of the six mutual incidence angles between the four light beams, constitute a set of ten quantities able to determine, in principle, the ten gravitational potentials. The study of these light-like coordinate systems opens the way towards an electro-magnetic method for the measure of the gravitational field.

c) Most of the problems posed by this type of coordinates (existence, multiplicity, characterization, etc.) are still open. Here we present first results, relatively simple, but that anticipate the vast range of applications that these coordinate systems may have in Relativity. In order to not overload the presentation, we have omitted the proofs. A detailed article, including them, is in preparation.

2 LIGHT COORDINATES. GENERAL PROPERTIES

a) For reasons of formalism, which will be more clear in Section 3, it is convenient not to fix the dimension of the space-time. We take thus a space-time (V_n, g) of dimension $n \geq 2$, and lorentzian metric g. In it, a beam of light is, by definition, a congruence of light-like geodesics[7]. Denoting by ξ a vector field of tangent directions, the light-like and geodesic character of its integral curves is defined respectively by the following relations,

$$g(\xi, \xi) = 0 \quad , \quad D_\xi \xi \wedge \xi = 0$$

where D_ξ is the directional derivation along ξ, $D_\xi \equiv i(\xi)\nabla$, and "$\wedge$" denotes the exterior product.

Consider now n beams of light, and let us denote by ξ_A $(A = 1, ..., n)$ a field of tangent directions to the A-th beam. In addition to the precedent relations, these fields must verify those insuring that the set $\{\xi_A\}$ constitute a natural frame, namely that the ξ_A commute; putting $D_A \equiv D_{\xi_A}$, one has:

In a space-time (V_n, g), light coordinate systems are defined by n vector fields ξ_A $(A = 1, ..., n)$ verifying the fundamental system:

$$\mathbf{I} : \begin{cases} g(\xi_A, \xi_A) = 0 \\ D_A \xi_A \wedge \xi_A = 0 \\ [\xi_A, \xi_B] = 0 \end{cases}$$

This system consists of $n + n(n-1) + \binom{n}{2} = n(3n-1)/2$ equations in n^2 unknowns ξ_A, so that, *for every $n \geq 2$ the fundamental system* $\mathbf{I}$ *is overdetermined.*

b) The third group of equations in $\mathbf{I}$ define over the ξ_A a commutative Lie algebra structure, completely independent of the metric structure of V_n. Consequently, this group is obviously invariant under conformal transformations of the metric; the first group in $\mathbf{I}$ is trivially invariant and, because of it, so is the second group. We have thus the important property:

The fundamental system $\mathbf{I}$ *is invariant under conformal transformations of the metric.*

It results that its study depends only on the conformal structure $(V_n, \{g\})$ associated to the given metric structure (V_n, g). The space of solutions of system **I** is thus independent of the representative g_r of the conformal class $\{g\}$ associated to g. Nevertheless, as we will see in paragraph **d)**, to every solution $\{\xi_A\}$ of **I** corresponds canonically a unique representative $\hat{g}$ of $\{g\}$.

c) Consider other vector fields of tangent directions to the geodesic beams: $\xi'_A = \lambda_A \xi_A$. The first two groups of equations in **I** are invariants, but the invariance of the third one imposes $L(\xi_{\bar{A}})\lambda_A = 0$, with $\bar{A} \neq A$; one has thus:

The local parameterization of the coordinate hypersurfaces corresponding to system **I** *is arbitrary.*

This property is very interesting from the physical point of view: it says us that, for the realization of light coordinate systems, we can choose arbitrarily the frequency of every beam, only its (transverse) gradient (phase) is determined by the geometry.

d) The second group of **I** is obviously equivalent to:

$$D_A \xi_A = \phi_A \xi_A \;, \tag{1}$$

where every ϕ_A $(A = 1, ..., n)$ is a function related to the non affine parameterization of the integral curves of ξ_A. Under the above two invariance properties, it is not possible to simultaneously annul the n functions ϕ_A. The following result, nevertheless, allow it:

The fundamental system **I** *is strictly equivalent to the system:*

$$\mathbf{I}' : \begin{cases} g(\xi_A, \xi_A) = 0 \\ L(\xi_A)\{\phi g(\xi_A, \xi_A)\} = 0 \\ [\xi_A, \xi_B] = 0 \end{cases}$$

In other words, under the first and third groups of equations in **I**, the ϕ_A in Eq.(1) are, up to a sign, necessarily of the form $L(\xi_A)\phi$. Consequently:

For every solution $\{\xi_A\}$ *of the fundamental system* **I**, *there exists a representative* $\hat{g} \equiv \phi g$ *of the conformal structure with respect to which the integral curves of all the* ξ_A *are simultaneously affine parameterized:* $D_A \xi_A = 0$.

Such a representative $\hat{g}$ will be called the *affine representative* of the solution $\{\xi_A\}$.

e) The nonlinear and overdetermined character of the system **I** makes difficult to obtain explicitly its general solution. In such a case and in order to shape out as well as possible the form of this general solution, one can, either to study in depth the general properties of the solutions, or to study and to group conveniently particular classes of solutions. First results in this second way are presented in the following Section, and are restricted essentially to the conformal structure associated to Minkowski space-time. In the first of these directions, we are studying two properties of great operational importance: involution and dimensional freedom.

- The involution property is related to the following operational problem: suppose that, at an initial instant, we have been able to meet in a region four light beams such that they constitute a light coordinate system; in the influence domain of this region (causal future), will the four beams continue to constitute a light coordinate system? The answer is *yes* if the system is *in involution* and *generically no* if it is *not in involution*.

- The dimensional freedom is related to the following experimental situation. Suppose we are in a space-time admitting light coordinate systems. Every time we find two light beams that define light coordinates, is it always possible to find a third light beam such that the three beams define three light coordinates? and, if this is possible, do will exist a fourth light beam that define with the above three four light coordinates, and so on? This will be possible iff the system is *completely free*.

From a mathematical point of view, if the fundamental system is in involution, it can be grouped in two subsystems, respectively of $n(n-1)/2$ and n^2 equations, such that the second subsystem (i) propagate the first subsystem and (ii) constitute a non degenerate evolution system in the Cauchy sense. And in order that the system be completely free, it is necessary that any solution corresponding to $A \leq n' \leq n-1$ could be completed to a solution corresponding to $n'' = n' + 1$. At the present moment, we do not know[8] neither if $\mathbf{I}$ is in involution nor if it is completely free.

3 LIGHT COORDINATES. PARTICULAR CASES

a) Let us begin with dimension 2. Although trivial, it is interesting in the case of decomposable or almost decomposable space-times, as well as in the study of dimensional freedom in arbitrary dimension.

Our study being local, in dimension 2 there exists a unique conformal structure $(V_2, \{\eta\})$, and this structure admits as a representative the lorentzian flat metric η. There exist only two light-like geodesic directions, which necessarily commute for suitable parameterizations: they constitute the sole solution to the system **I** . From the second proposition of paragraph d) of the previous Section, it follows that, up to homothecy, *the sole affine representative of the two-dimensional conformal structure is flat metric* η.

For every two-dimensional metric g, there exists thus one, and only one, light coordinate system. In this system, say $\{\tau, s\}$, the metric has the form:

$$g = f(\tau, s)\tau \quad \textit{with} \quad \eta = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{2}$$

where, of course, f is the conformally flat factor.

b) Let us consider the three-dimensional case. As a consequence of the form $\mathbf{I}'$ of the fundamental system, the affine representative $\hat{g}$ of every solution to **I** is, in local charts adapted to the solution, of the form:

$$\hat{g} = \begin{pmatrix} 0 & \alpha(x^3) & \gamma(x^2) \\ \alpha(x^3) & 0 & \beta(x^1) \\ \gamma(x^2) & \beta(x^1) & 0 \end{pmatrix} \tag{3}$$

Its signature is ± 1 according to the sign $\mp$ of the quantity $\alpha\beta\gamma$. From (3) it may be shown:

The necessary and sufficient conditions for the affine representative $\hat{g}$ *of a three-dimensional light coordinate system to be flat, is that its components* (3) *be constants.*

Let us note that this result does not implies that the only light coordinate systems admitted by a three-dimensional minkowskian metric be the above mentioned: for particular, non constant, values of α, β and γ in (3), $\hat{g}$ could be conformally flat. On the other hand, although we do not know if all three-dimensional metrics admit light coordinates, we know that *some* of them do *not* admit affine parameterized light coordinates:

The fundamental system **I** *does not admit metrics of regular Einstein spaces (of non vanishing scalar curvature) as affine parameterized representative metrics.*

c) Finally consider the case of physical space-time ($n = 4$). From the form $\mathbf{I}'$ of the fundamental system, it follows now that the affine representative $\hat{g}$

of every solution of **I** is, in local charts adapted to the solution, of the form:

$$\hat{g} = \begin{pmatrix} 0 & \alpha(x^2, x^3) & \gamma(x^1, x^3) & \epsilon(x^1, x^2) \\ \alpha(x^2, x^3) & 0 & \phi(x^0, x^3) & \delta(x^0, x^2) \\ \gamma(x^1, x^3) & \phi(x^0, x^3) & 0 & \beta(x^0, x^1) \\ \epsilon(x^1, x^2) & \delta(x^0, x^2) & \beta(x^0, x^1) & 0 \end{pmatrix} \tag{4}$$

Every $\hat{g}$ of the form (4) has necessarily hyperbolic signature, but *it is lorentzian only if* $\det \hat{g} < 0$. Then (i) *all the extra-diagonal components are necessarily different from zero,* (ii) *the cofactors of the diagonal elements have same sign,* and (iii) *if this sign is* ϵ, the signature of $\hat{g}$ is -2ϵ.

Analogously to the cases $n = 2$ and $n = 3$, it can also be shown:

The necessary and sufficient condition for the affine representative $\hat{g}$*, of a four-dimensional light coordinate system to be flat is that its components* (4) *be all constant.*

Although we suspect that this result is independent of the dimension, we have not yet found a proof valid for any n; this is why the statement is included here, and not in the preceding Section. Again, let us remark that this proposition does *not* implicate that the flat minkowskian metric η only admit as light coordinates those for which the metric components are constant, but that all the other light coordinates admitted by η are not affine parameterized.

The following result is of great conceptual and practical importance. It not only allows to easily obtain non affine parameterized light coordinates for η, but is the key for the study of the relation between the existence of light coordinates and Petrov-Bel algebraic types of space-times:

The necessary and sufficient condition for one of the vector fields solution of the system **I** *to be shear-free, is that the mutual quotients between the extra-diagonal components of* $\hat{g}$ *in the complementary directions of the field, be invariant for the vector field.*

Thus, for example, if we enumerate the vector fields ξ_A in such a way that the shear-free one be ξ_0, it follows from (4) that the quotients β/δ and β/ϕ are independent of the coordinate x^0, that is to say, that there exist functions of one variable, $\lambda_A(x^A)$ such that $\beta = \lambda_0\lambda_1$, $\delta = \lambda_0\lambda_2$, $\phi = \lambda_0\lambda_3$. According to Goldberg-Sachs theorem, non conformally flat space-times do not admit more than two shear-free geodetic null congruences. For this reason, all the space-times for which the above relations take place for three or more coordinate are necessarily conformally flat, and the corresponding coordinates constitute a non affine parameterized light coordinate system of the minkowskian flat metric η.

As an example, we consider here the extreme case:

Let $\lambda_A(x^A)$ be four functions of one variable, and P, Q and R three constants such that

$$P^2 + Q^2 + R^2 - 2(PQ + QR + RP) \neq O \ .$$

Then the metric

$$\hat{g} = \begin{pmatrix} 0 & P\lambda_2\lambda_3 & Q\lambda_1\lambda_3 & R\lambda_1\lambda_2 \\ P\lambda_2\lambda_3 & 0 & \lambda_0\lambda_3 & \lambda_0\lambda_2 \\ Q\lambda_1\lambda_3 & \lambda_0\lambda_3 & 0 & \lambda_0\lambda_1 \\ R\lambda_1\lambda_2 & \lambda_0\lambda_2 & \lambda_0\lambda_1 & 0 \end{pmatrix}$$

is conformally flat and the $\{x^A\}$ constitute a non affine parameterized light coordinate system for η, whose four light-like directions are shear-free.

References

[1] *Classical Physics* is used here as an analog of Newtonian Physics, as opposed to Relativistic Physics. In any case we are not speaking here of Quantum Physics.

[2] *Extension* here means substitution of the (homothetic) dimensional invariance group by the (local) diffeomorphism (pseudo-)group of $\Re^3$ (coordinate charts).

[3] It is obvious because the invariance under the general (pseudo-)group of (local) diffeomorphisms of $\Re^4$ implies *a fortiori* local covariance.

[4] For example, the theorical or experimental confirmation that a specific physical field, under particular conditions, depends only of the variable r is void or, at least, confuse if the nature of the coordinate lines "r variable" is not precised (radial, cilyndrical, angular or other). And, from the experimental point of view, we need to describe the physical procedure for its construction. It is this sense that has, for us here, the word *operational.*

[5] The difficulties of the operational definition of cartesian coordinate systems (or others) in Relativity are related to that of the physical construction of their coordinate lines: being space-like, they would need to be materialized by means of tachyons.

[6] Typically, the definition domains of such systems correspond to world tubes obtained by evolution of space-like tetrahedric figures, over whose four faces, at every instant, light beams fall on.

[7] Here we consider light in the geometric optics approximation, that is to say, as a fluid of point like "luxons".

[8] In fact, we know that the system **I** is, strictly speaking, *non involutive.* But from the physical point of view what has interest is the possibility to associate to it an *equivalent* involutive evolution system; it is the existence of this system that is open.