OPTICAL COORDINATE SYSTEM FOR A LOCAL OBSERVER IN A WEAK GRAVITATIONAL FIELD

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ABSTRACT. We develop a method to construct the reference system (RS) of a local observer which is based on the transformation from the instant normal coordinates to the Fermi and optical ones. The main advantage of our approach is due to a direct relation of the optical coordinates with observable positions of distant objects on the celestial sphere. The method is applied to a construction of local observer RS in a weak field within the linearized gravitation theory.

1. INTRODUCTION

Outstanding perspectives of microarcsecond astrometry connected with challenging space projects such as GAIA mission, demand to construct the reference systems (RS) which would be accurate enough to deal with fine relativistic effects. On the other hand these RS must be convenient and clearly understandable as much as possible. The IAU Resolutions concerning the reference frames, adopted by the XXIVth International Astronomical Union General Assembly focus mainly on harmonic coordinates, and this direction is most developed (see Soffel et al., 2003). Harmonic coordinates are convenient to be used in the barycentric system for limited ensemble of masses with asymptotically flat metric and the space origin at the barycentre of the ensemble of bodies. In this barycentric system we can easily fix the coordinates if they are quasi-Cartesian and the metric tensor is pseudo-Euclidian at the infinity. We remind that the harmonicity conditions have the form of partial differential equations and for unique determination of harmonic coordinates additional conditions are required. For the case of geocentric coordinates or coordinates associated with a satellite this choice looks something artificial. It is not linked with any physical or geometrical preferences, but just with the method of solving the Einstein equations for metric tensor or with the particular choice of transformations to the barycentric system. Also, a genetic relationship of harmonic coordinates with Einstein's equation requires additional efforts (see, e.g., Klioner & Soffel, 2000; Klioner, 2003) when the General Relativity is compared with alternative theories of gravity.

Harmonic coordinates are not observable and are not associated with the observations. Therefore, their use does not solve directly the problem of the interpretation of observations. On the other hand, there are the well known relativistic reference frames that are based on invariant interrelations characterizing the observables. These RS are thus connected with results of observations in a direct way. Such relations are determined correctly for any kind of metric, despite the kind of field equations, and they are independent on a physical model of the reference body. As an example of such systems we can remind the local observer's frame, based on the Riemannian normal coordinates (RNC), Fermi coordinates (FC) or on optical ones (OC) (Synge, 1960). Reference systems of the local observer are based on the geodesics lines and have a clear geometric interpretation. The Fermi coordinates are the most direct relativistic generalization of the reference frame of the moving observer in Newtonian mechanics. At the same time, the optical coordinates that operate directly with the position of an object on the celestial sphere, are most closely associated with the observations. Transition to the optical coordinates can be treated as an important step in solving the problem of observables. Fermi coordinates are better known (see e.g. Ashby & Bertotti, 1986; Fukushima, 1988; Aleksandrov et al., 1990; Marzlin, 1994; Nesterov A. 1999), while the optical coordinates have not been given due attention. However, in recent years some version of optical coordinates is used in cosmology under the name "observational coordinates" (Clarkson & Maartens, 2010). Note also a possibility of introducing a generalized EC and OC, measured from the surface of the Earth, instead from its center (Zhdanov, 1994).

Here we demonstrate how the developed mathematical apparatus associated with geodesics, their deviation, and parallel transport is used to construct coordinate transformations to FC and OC, and to find the metric in these coordinates for an arbitrary weak field. Note the ideological affinity of our approach to work by Nesterov (1999). However, we achieve a significant simplification by transition from integration of the curvature tensor to the integrals of the metric perturbations.

2. BASIC RELATIONS IN GENERAL

Suppose that the observer is moving along the world line $x_c^\mu(\tau)$, τ is his proper time, and $e^\alpha_{(\mu)}$ is his proper reference frame (here, the index in parentheses indicates the number of the vector). Vector $e^\alpha_{(0)}$ coincides with observer's four-velocity, i.e. $\frac{dx_c^\alpha}{d\tau} = u^\alpha = e^\alpha_{(0)}$. Proper frame is transported along the observer's world line as follows (e.g. Misner, Thorne and Wheeler, 1973):

$$\frac{De^\alpha_{(\mu)}}{\partial\tau} = \Omega^\alpha{}_\beta e^\beta_{(\mu)}. \tag{1}$$

Here $\Omega_{\alpha\beta} = a_\alpha u_\beta - u_\alpha a_\beta + \varepsilon_{\alpha\beta\gamma\delta} u^\gamma \omega^\delta$ is the four-tensor of observer's rotation; a^α his four-acceleration, ω^β angular velocity.

Consider geodesic $x^\mu(\tau, s)$ parameterized with canonical parameter s, which passes through the observed point and $x^\mu(\tau, 0) = x_c^\mu(\tau)$. Let v^α be a tangential ort to this geodesic at the point $x_c^\mu(\tau)$. Then (instant) RNK of the point $x^\mu(\tau, s)$ adapted to the tetrad $e^\alpha_{(\mu)}$ and originated at $x_c^\mu(\tau)$ are $y^\mu = v^\mu s$, where $v^\mu = e^{(\mu)}_\alpha v^\alpha$. So, to find formulae for the transformation to normal coordinates, one needs to construct the general solution $x^\mu(\tau, s) = X^\mu(x_c^\nu(\tau), v^\alpha s)$ of Cauchy problem for the geodesic equation

$$\frac{d^2x^\mu}{ds^2} + \Gamma^\mu_{\nu\lambda}\frac{dx^\nu}{ds}\frac{dx^\lambda}{ds} = 0. \tag{2}$$

In the case of an analytic metric this solution is known in a form of the covariant Taylor series (see Pyragas et al., 1995). Also, in the weak-field approximation it is easy to present the solution in the integral form (see below).

The construction of Fermi coordinates involves only the geodesics, which are orthogonal to the world line of the observer $g_{\alpha\beta}v^\alpha u^\beta = v^0 = 0$. Then, FC z^μ are defined by the following relations:

$$z^0 = \tau, \quad z^i = y^i, \quad i = 1, 2, 3. \tag{3}$$

Similarly, the optical coordinates ζ^μ are constructed by means of light geodesics $g_{\alpha\beta}v^\alpha v^\beta = 0$ of the past $v^0 = -\sqrt{\sum_{i=1}^3 (v^i)^2}$:

$$\zeta^0 = \tau, \quad \zeta^i = y^i. \tag{4}$$

Jacobi matrices, which connect the tensor components in the FC (or OC) and RNC include solutions of the equation of geodesic deviation. The corresponding general formulae were found by Zhdanov and Alexandrov (1990) and Alexandrov and Zhdanov (1992) (see also Pyragas et al., 1995). A fundamental role is played by the matrices $S^\rho_\sigma(y^\mu)$ and $C^\rho_\sigma(y^\mu)$, which satisfy the equations:

$$D^2\mathbf{S} + D\mathbf{S} = \tilde{\mathbf{r}}\mathbf{S}, \tag{5}$$

$$D^2\mathbf{C} - D\mathbf{C} = \tilde{\mathbf{r}}\mathbf{C}. \tag{6}$$

Here $D = y^\mu \frac{\partial}{\partial y^\mu}$, $\tilde{r}^\rho_\sigma = \tilde{R}^\rho_{\mu\nu\sigma}(y^\tau) y^\mu y^\nu$, $\tilde{R}^\rho_{\mu\nu\sigma}$ is a result of the parallel transport of the curvature tensor along the geodesic to the reference point. Geodesic deviations and the metric tensor in normal coordinates as well as the aforementioned Jacobi matrixes are expressed through these matrices. In particular, for the metric tensor in optical coordinates $g^{Opt}_{\mu\nu}$ we have (Alexandrov & Zhdanov, 1992)

$$g^{opt}_{\mu\nu} = \eta_{\rho\sigma} G^{O\rho}_\mu G^{O\sigma}_\nu, \tag{7}$$

where

$$G^{O\rho}_0 = C^\rho{}_0 + S^\rho{}_i \Omega^i{}_k y^k, \quad G^{O\rho}_j = S^\rho{}_j + S^\rho{}_0 \frac{y^j}{y^0}. \tag{8}$$

Similarly, for the metric in Fermi coordinates $g_{\mu\nu}^{Fermi}$ (Zhdanov & Alexandrov, 1990)

$$g_{\mu\nu}^{Fermi} = \eta_{\rho\sigma} G_{\mu}^{F\rho} G_{\nu}^{F\sigma}, \tag{9}$$

$$G_{0}^{F\rho} = C^{\rho}{}_{0} + S^{\rho}{}_{i} \Omega^{i}{}_{k} y^{k}, \quad G_{j}^{F\rho} = S^{\rho}{}_{j}. \tag{10}$$

It should be noted that the matrixes $G_{\mu}^{O\rho}$ and $G_{\mu}^{F\rho}$ appearing here are nothing but the operator of parallel transport along the geodesic in the corresponding coordinates. Thus, $G_{\mu}^{O\rho}$ directly describes the transfer of the wave 4-vector and the polarization from the source to the observer.

3. WEAK FIELD

In weak-field approximation the metric tensor of spacetime $g_{\mu\nu}(x^{\tau})$ is treated as a sum of Minkowski tensor $\eta_{\mu\nu}$ and a perturbation term $h_{\mu\nu}$, so that

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x^{\tau}), \tag{11}$$

where all of the components of $h_{\mu\nu}$ are much less than one (and similarly for all derivatives of $h_{\mu\nu}$). Then one ignores all products of $h_{\mu\nu}$ (or its derivatives). The Christoffel symbols can be calculated as

$$\Gamma_{\nu\lambda}^{\mu} = \frac{1}{2} g^{\mu\tau} \left(g_{\tau\nu,\lambda} + g_{\tau\lambda,\nu} - g_{\nu\lambda,\tau} \right) = \frac{1}{2} \eta^{\mu\tau} \left(h_{\tau\nu,\lambda} + h_{\tau\lambda,\nu} - h_{\nu\lambda,\tau} \right), \tag{12}$$

and Riemann tensor as

$$R_{\rho\mu\nu\sigma} = \frac{1}{2} \left(h_{\rho\sigma,\mu\nu} + h_{\mu\nu,\rho\sigma} - h_{\mu\sigma,\nu\rho} - h_{\nu\rho,\mu\sigma} \right). \tag{13}$$

A simple coordinate transformation of the form

$$x^{\mu} = x'^{\mu} - \frac{1}{2} h_{\nu}^{\mu}(x_{c}^{\tau}) (x'^{\nu} - x_{c}^{\nu}) - \frac{1}{2} \Gamma_{\nu\sigma}^{\mu}(x_{c}^{\tau}) (x'^{\nu} - x_{c}^{\nu}) (x'^{\sigma} - x_{c}^{\sigma}) \tag{14}$$

converts metric tensor as follows

$$g'_{\mu\nu}(x'^{\tau}) = \eta_{\mu\nu} + h_{\mu\nu}(x'^{\tau}) - h_{\mu\nu}(x_{c}^{\tau}) - h_{\mu\nu,\tau}(x_{c}^{\tau}) (x'^{\tau} - x_{c}^{\tau}). \tag{15}$$

In small terms differences $(x'^{\tau} - x_{c}^{\tau})$ are replaced with y^{τ}:

$$g'_{\mu\nu}(x'^{\tau}) = \eta_{\mu\nu} + h'_{\mu\nu}(x_{c}^{\tau}, y^{\mu}), \tag{16}$$

$$h'_{\mu\nu}(x_{c}^{\tau}, y^{\mu}) = h_{\mu\nu}(x_{c}^{\tau} + y^{\mu}) - h_{\mu\nu}(x_{c}^{\tau}) - h_{\mu\nu,\sigma}(x_{c}^{\tau}) y^{\sigma}. \tag{17}$$

In the case of moving reference point $x_{c}^{\mu}(\tau)$ this transformation depends on the parameter τ.

The transformed metric satisfies the following conditions:

$$g'_{\mu\nu}(x_{c}^{\tau}) = \eta_{\mu\nu}, \quad g'_{\mu\nu,\tau}(x_{c}^{\sigma}) = \Gamma'_{\mu\nu,\tau}(x_{c}^{\sigma}) = 0.$$

In order to simplify the formulae below, we shall omit the primes associated with the transformation (12).

Let's introduce three sets of integrals through which all necessary quantities can be expressed:

$$I_{\mu\nu}(x_{c}^{\sigma}, y^{\tau}) = \frac{1}{s} \int_{0}^{s} h_{\mu\nu}(x_{c}^{\sigma}, s_{1} v^{\sigma}) \, ds_{1}, \quad J_{\mu\nu}(x_{c}^{\sigma}, y^{\tau}) = \int_{0}^{s} \frac{h_{\mu\nu}(x_{c}^{\sigma}, s_{1} v^{\sigma})}{s_{1}} ds_{1},$$

$$K_{\mu\nu}(x_{c}^{\sigma}, y^{\tau}) = s \int_{0}^{s} \frac{h_{\mu\nu}(x_{c}^{\sigma}, s_{1} v^{\sigma}) \, ds_{1}}{s_{1}^{2}}. \tag{18}$$

Integrating the geodesic equation (2) with the expression (12), we find the transformation to RNC (cf. Marzlin, 1994)

$$x^\mu = x_c^\mu + y^\mu - y^\nu \eta^{\mu\sigma} I_{\nu\sigma} + \frac{1}{2}\eta^{\mu\sigma} y^\nu y^\lambda \left(J_{\nu\lambda} - I_{\nu\lambda}\right)_{,\sigma}. \tag{19}$$

There and below the comma denotes the partial derivative with respect to the normal coordinates. In the case of a weak field matrices $S^\rho_\sigma(y^\mu)$ and $C^\rho_\sigma(y^\mu)$ are presented in the following form:

$$S^\rho_\sigma = \delta^\rho_\sigma + \Sigma^\rho_\sigma, \quad C^\rho_\sigma = \delta^\rho_\sigma + \Delta^\rho_\sigma, \tag{20}$$

Σ, Δ being small. We linearize equations (5,6) and by integration we obtain

$$\Sigma_{\mu\nu} = \frac{1}{2}\left[h_{\mu\nu} - 2I_{\mu\nu} + 2y^\rho\left(J_{\rho(\nu,\mu)} - 2I_{\rho(\mu,\nu)}\right) + y^\rho y^\sigma\left(J_{\rho\sigma,\mu\nu} - I_{\rho\sigma,\mu\nu}\right)\right], \tag{21}$$

$$\Delta_{\mu\nu} = \frac{1}{2}\left[h_{\mu\nu} - 2y^\rho J_{\rho(\mu,\nu)} + y^\rho y^\sigma\left(K_{\rho\sigma,\mu\nu} - J_{\rho\sigma,\mu\nu}\right)\right]. \tag{22}$$

Substituting these expressions into (7-10) one can easily obtain expressions for the metric in FC and OC. Of course, the same expressions can be obtained by successive transformation of coordinates first to RNC (19) and then to the FC and OC in accordance with formulae (3) and (3).

4. SUMMARY

We developed the method to construct the reference frame of a local observer basing on geodesics, their deviation, and parallel transport. We applied it to the non-inertial observer in an arbitrary weak gravitational field, for two kinds of coordinates: optical and Fermi. We have found the transformation formulae to the new systems and the metric tensor components in the new systems via the integrals of the metric perturbations. The results can be useful for the interpretation of the most precise astrometric projects such as GAIA or future space VLBI.

5. REFERENCES

Aleksandrov A.N., Zhdanov V.I., Parnovskii S.L., 1990, "Relativistic frame of reference near the Erth and radiointerferometric observations", Kinematica i Fizika Nebesnykh Tel, **6**, pp. 3-7

Alexandrov A.N., Zhdanov V.I., 1992, "To the theory of reference frames based on the optical coordinates", Herald Taras Shevchenko Nat. Univ. Kyiv, Phys.-Math., **3**, pp. 6-11 (in Ukrainian)

Ashby N., Bertotti B. 1986, "Relativistic Effects in Local Inertial Frames", Phys. Rev. D. **34**(8), pp. 2246-2259

Clarkson C., Maartens R., 2010, "Inhomogeneity and the foundations of concordance cosmology", Class. Quantum Grav., **27**, 124008

Fukushima T. 1988, "The Fermi coordinates for weak gravitational fields", Cel. Mech., **44**, pp. 61-75

Klioner S.A., 2003. "A Practical Relativistic Model for Microarcsecond Astrometry in Space", AJ, **125** pp. 1580-1597

Klioner S.A., Soffel M.H. 2000, "Relativistic celestial mechanics with PPN parameters", Phys. Rev. D **62**, id.024019.

Marzlin K., 1994, "Fermi coordinates for a weak gravitational fields", Phys. Rev., D **50**, pp. 888-891

Misner C.W., Thorne K.S., Wheeler J.A., 1973, Gravitation, Freeman and Co, San Francisco

Nesterov A. 1999, "Riemann normal coordinates, Fermi reference system and the geodesic deviation equation" Class. Quantum. Grav., **16**, pp.465-477

Pyragas K.A., Zhdanov V.I., Alexandrov A.N., Kudrya Yu.N., Pyragas L.E., 1995, Qualitative and analytical methods in relativistic dynamics, Energoatomizdat, Moscow (in Russian)

Soffel M., Klioner S., Petit G., et al., 2003, "The IAU 2000 resolutions for astrometry, celestial mechanics and metrology in the relativistic framework: explanatory supplement", Astronomical Journal, **126,** pp. 2687-2706.

Synge J.L., 1963, General relativity, North-Holland, Amsterdam

Zhdanov V.I., 1994, "Relativistic models of reference systems in the vicinity of the Earth" In: Studying Earth as a planet by methods of astronomy, geophysics and geodesy, Ya.S. Yatskiv (ed.), MAO, Kiev, pp. 41-53 (in Russian)

Zhdanov V.I., Alexandrov A.N., 1990, "Fermi coordinates and radiointerferometric observations", Herald Taras Shevchenko Nat. Univ. Kyiv , Astronomy, **32**, pp. 24-28 (in Russian)