TAKING INTO ACCOUNT THE PLANETARY PERTURBATIONS IN THE MOON'S THEORY

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ABSTRACT. The semi–analytical Moon's theory is treated in the form compatible with the general planetary theory GPT (Brumberg, 1995). The Moon is considered to be an additional planet in the field of eight major planets. Hence, according to the technique of the GPT, the theory of the orbital lunar motion can be presented by means of the series in the evolutionary eccentric and oblique variables with quasi-periodic coefficients in mean longitudes of the planets and the Moon. The time dependence of the evolutionary variables is determined by the trigonometric solution of the autonomous secular system describing the secular motions of the lunar perigee and node with taking into account the secular planetary inequalities. In this paper the right–hand members of the secular system are obtained in the analytical form. All the analytical calculations are performed by the echeloned Poisson series processor EPSP (Ivanova, 2001).

1. EQUATIONS OF LUNAR MOTION

The equations of the Moon's motion in geocentric rectangular coordinates x, y, z are described in the classical form

$$\ddot{x} = \frac{\partial U}{\partial x}\,, \qquad \ddot{y} = \frac{\partial U}{\partial y}\,, \qquad \ddot{z} = \frac{\partial U}{\partial z} \tag{1}$$

with the force function U

$$U = n^2\,\mathrm{a}^2\left[\frac{\mathrm{a}}{r} + \left(\frac{n_3}{n}\right)^2 \frac{M_s}{M_s+M_3} \sum_{k=1}^{\infty} A_3^{(k-1)} \left(\frac{r}{\mathrm{a}}\right)^{k+1} \left(\frac{\mathrm{a}_3}{r_3}\right)^{k+2} P_{k+1}\left(\omega_3\right)\right.$$

$$\left. + \sum_{i=1, i\neq 3}^{8} \left(\frac{n_i}{n}\right)^2 \frac{M_i}{M_s+M_i} \sum_{k=1}^{\infty} A_i^{(k-1)} \left(\frac{r}{\mathrm{a}}\right)^{k+1} \left(\frac{\mathrm{a}_i}{\Delta_{3i}}\right)^{k+2} P_{k+1}\left(\omega_i\right)\right]. \tag{2}$$

$$M_3 = M_e + M_m, \quad A_i^{(k-1)} = \epsilon_k \left(\frac{\mathrm{a}}{\mathrm{a}_i}\right)^{k-1}, \quad \epsilon_k = \left(\frac{M_m}{M_3}\right)^k + (-1)^{k+1}\left(\frac{M_e}{M_3}\right)^k,$$

$$\Delta_{3i} = \mid \mathbf{r_3} - \mathbf{r_i} \mid, \qquad \omega_3 = \frac{\mathbf{r r_3}}{r\, r_3}, \qquad \omega_i = \frac{\mathbf{r(r_3 - r_i)}}{r\,\Delta_{3i}}\,.$$

Here M_s, M_e, M_m are the masses of the Sun, the Earth and the Moon, respectively. The index i points to the principal planet with number i. a, n, $\mathbf{r}$ are the semi–major axis, mean motion and radius–vector of the Moon, respectively. a_i, n_i, $\mathbf{r_i}$ $(i = 1, 2, \ldots, 8)$ are the same for the major planets. $P_k(\omega_i)$ are the Legendre polynomials. For the Moon, the values both with index 9 and without any indices are used.

Instead of rectangular coordinates $\mathbf{r} = (x, y, z)$ one introduces the dimentionless complex conjugate variables p, q and real variable w, representing deviations from the planar circular motion

$$x + \sqrt{-1}\,y = \mathrm{a}(1-p)\exp\sqrt{-1}\,\lambda, \quad q = \overline{p}, \quad z = \mathrm{a}\,w, \quad w = \overline{w}\,, \tag{3}$$

λ being the mean longitude of the Moon. Here the bar means a conjugate value. The heliocentric coordinates $\mathbf{r_i} = (x_i, y_i, z_i)$ of the principal planets are subjected to the similar transformation.

In terms of the new variables the equations of lunar motion take the form

$$\ddot{p} + 2\sqrt{-1}n\dot{p} - \frac{3}{2}n^2(p+q) = n^2 P\,,$$

$$\ddot{w} + n^2 w \qquad = n^2 W \tag{4}$$

with the right–hand members

$$P = -1 - \frac{1}{2}p - \frac{3}{2}q + \frac{2}{n^2\,\mathrm{a}^2}\frac{\partial U}{\partial q}, \qquad W = w + \frac{1}{n^2\,\mathrm{a}^2}\frac{\partial U}{\partial w}\,. \tag{5}$$

2. RIGHT–HAND MEMBERS

The right–hand members can be expressed in the form

$$P = P^{(kep)} + P^{(sol)} + P^{(pla)}\,, \qquad W = W^{(kep)} + W^{(sol)} + W^{(pla)}\,. \tag{6}$$

Here P^{kep} and W^{kep} correspond to the keplerian geocentric lunar motion:

$$P^{kep} = -1 - \frac{1}{2}p - \frac{3}{2}q + (1-p)\frac{\mathrm{a}^3}{r^3}, \qquad W^{kep} = \left(1 - \frac{\mathrm{a}^3}{r^3}\right) w\,. \tag{7}$$

P^{sol} and W^{sol} are due to the action of the Sun:

$$P^{(sol)} = \left(\frac{n_3}{n}\right)^2 \frac{M_s}{M_s + M_3} \sum_{k=1}^{\infty} \epsilon_k \left(\frac{\mathrm{a}}{\mathrm{a}_3}\right)^{k-1} P_k{}^{(sol)}\,,$$

$$P_k{}^{(sol)} = (1-p)\left(\frac{r}{\mathrm{a}}\right)^{k-1}\left(\frac{\mathrm{a}_3}{r_3}\right)^{k+2} C_{k-1}^{\frac{3}{2}}(\omega_3) - (1-p_3)\,\zeta_3^{-1}\left(\frac{r}{\mathrm{a}}\right)^{k}\left(\frac{\mathrm{a}_3}{r_3}\right)^{k+3} C_k^{\frac{3}{2}}(\omega_3)\,, \tag{8}$$

$$W^{(sol)} = \left(\frac{n_3}{n}\right)^2 \frac{M_s}{M_s + M_3} \sum_{k=1}^{\infty} \epsilon_k \left(\frac{\mathrm{a}}{\mathrm{a}_3}\right)^{k-1} W_k{}^{(sol)}\,,$$

$$W_k{}^{(sol)} = -w\left(\frac{r}{\mathrm{a}}\right)^{k-1}\left(\frac{\mathrm{a}_3}{r_3}\right)^{k+2} C_{k-1}^{\frac{3}{2}}(\omega_3) + w_3\left(\frac{r}{\mathrm{a}}\right)^{k}\left(\frac{\mathrm{a}_3}{r_3}\right)^{k+3} C_k^{\frac{3}{2}}(\omega_3)\,. \tag{9}$$

P^{pla} and W^{pla} are responsible for the planetary perturbations:

$$P^{(pla)} = \sum_{i=1, i\neq 3}^{8} \left(\frac{n_i}{n}\right)^2 \frac{M_i}{M_s + M_i} \sum_{k=1}^{\infty} \epsilon_k \left(\frac{\mathrm{a}}{\mathrm{a}_i}\right)^{k-1} \left(\frac{\mathrm{a}_i}{\mathrm{a}_{3i}}\right)^{k+2} P_{ik}^{(pla)}\,,$$

$$P_{ik}{}^{(pla)} = (1-p)\left(\frac{r}{\mathrm{a}}\right)^{k-1}\left(\frac{\mathrm{a}_{3i}}{\Delta_{3i}}\right)^{k+2} C_{k-1}^{\frac{3}{2}}(\omega_i) -$$

$$-\left(\frac{r}{\mathrm{a}}\right)^{k}\left[(1-p_3)\,\zeta_3^{-1}\frac{\mathrm{a}_3}{\mathrm{a}_{3i}} - (1-p_i)\,\zeta_i^{-1}\frac{\mathrm{a}_i}{\mathrm{a}_{3i}}\right]\left(\frac{\mathrm{a}_{3i}}{\Delta_{3i}}\right)^{k+3} C_k^{\frac{3}{2}}(\omega_i)\,, \tag{10}$$

$$W^{(pla)} = \sum_{i=1, i\neq 3}^{8} \left(\frac{n_i}{n}\right)^2 \frac{M_i}{M_s + M_i} \sum_{k=1}^{\infty} \epsilon_k \left(\frac{\mathrm{a}}{\mathrm{a}_i}\right)^{k-1} \left(\frac{\mathrm{a}_i}{\mathrm{a}_{3i}}\right)^{k+2} W_{ik}^{(pla)}\,,$$

$$W_{ik}{}^{(pla)} = -w\left(\frac{r}{\mathrm{a}}\right)^{k-1}\left(\frac{\mathrm{a}_{3i}}{\Delta_{3i}}\right)^{k+2} C_{k-1}^{\frac{3}{2}}(\omega_i) + \left(\frac{r}{\mathrm{a}}\right)^{k}\left(w_3\frac{\mathrm{a}_3}{\mathrm{a}_{3i}} - w_i\frac{\mathrm{a}_i}{\mathrm{a}_{3i}}\right)\left(\frac{\mathrm{a}_{3i}}{\Delta_{3i}}\right)^{k+3} C_k^{\frac{3}{2}}(\omega_i)\,. \tag{11}$$

Here $\mathrm{a}_{3i} = \max\{\mathrm{a}_3, \mathrm{a}_i\}$, $\quad C_k^{\frac{3}{2}}(\omega_i)$ are Gegenbauer polynomials.

The right–hand members are expanded into the Poisson series in power and exponential variables

$$p,\ q,\ w,\ p_i,\ q_i,\ w_i,\ \zeta_i = \exp\sqrt{-1}\,(\lambda - \lambda_i), \qquad i = 1, 2, \ldots, 8. \qquad (12)\,.$$

The keplerian and solar right–hand members were obtained in (Ivanova, 2011) concerning the determination of the secular indirect planetary perturbations within the frames of the main problem.

In constructing the Moon's theory there is no necessity to have a very accurate theory of motion for the major planets. It is sufficient to use only Keplerian terms, a first–order intermediary and first–order linear theory. Therefore the coordinates of the major planets necessary for the Moon's theory may be presented in the form

$$p_i = \delta p_i^{(0)} + p_{i,0}^{(1)} + \delta p_{i,1}^{(1)}\,, \qquad w_i = w_i^{(0)} + w_{i,1}^{(1)} \tag{13}$$

where the upper indices point out the order of smallness relative to the masses. The second sub-indices are responsible for the orders of the eccentricities and inclinations.

The first parts stand here for the Keplerian terms determined by the literal expansions in powers of the complex Laplace–type variables $a_i, \overline{a}_i, b_i, \overline{b}_i$ propotional to the eccentricity and inclination of the body with number i

$$\delta p_i^{(0)} = \sum p_{klmn}^{(0)}\, a_i^k \overline{a}_i^l b_i^m \overline{b}_i^n\,, \qquad w_i^{(0)} = \sum w_{klmn}^{(0)}\, a_i^k \overline{a}_i^l b_i^m \overline{b}_i^n\,. \tag{14}$$

The coefficients in Equation (14) are numerical constants.

The second part in p_i describes the terms of first order relative to the mass parameter in the intermediate solution

$$p_{i,0}^{(1)} = \sum_{\gamma} p_{i,\gamma}^{(1)}\, \exp\sqrt{-1}\,(\gamma\lambda)\,,$$

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_8), \quad (\gamma\lambda) = \sum_{j=1}^{8} \gamma_j \lambda_j, \quad \sum_{j=1}^{8} \gamma_j = 0\,. \tag{15}$$

And the last parts contain the linear terms in the eccentricities and inclinations taking into account only the first order terms relative to the masses. In explicit form they are expressed by the relations

$$\delta p_{i,1}^{(1)} = \sum_{j=1, j\neq i}^{8} \delta p_{ij,1}^{(1)}, \qquad w_{i,1}^{(1)} = \sum_{j=1, j\neq i}^{8} w_{ij,1}^{(1)}\,, \tag{16}$$

$$\delta p_{ij,1}^{(1)} = c(i,j,0)a_i + d(i,j,0)\overline{a}_i + c(i,j,1)a_j + d(i,j,1)\overline{a}_j\,,$$

$$w_{ij,1}^{(1)} = f(i,j,0)b_i + \overline{f}(i,j,0)\overline{b}_i + f(i,j,1)b_j + \overline{f}(i,j,1)\overline{b}_j$$

with quasi–periodic coefficients c, d and f.
The right–hand members are obtained in a purely analytical form.

3. INTERMEDIARY

In accordance with the general planetary technique the solution of the Moon's equations is represented in the form

$$p = p^{(0)} + \delta p, \qquad w = \delta w \tag{17}$$

where

$$p = p^{(0)}, \qquad w = 0 \tag{18}$$

is a particular planar quasi-periodic solution provided that the major planets move in their intermediate orbits

$$p_i = p_i^{(0)}, \qquad w_i = 0\,.$$

This solution generalizes Hill's variational curve and includes all solar and planetary inequalities independent of the eccentricities and inclinations of all the bodies. The solution is represented by the multiple Fourier series

$$p^{(0)} = \sum_{\gamma} p_{\gamma}\, \exp\sqrt{-1}\,(\gamma\lambda)\,,$$

$$\gamma = (\gamma_1, \gamma_2, \dots, \gamma_9), \quad (\gamma\lambda) = \sum_{i=1}^{9} \gamma_i \lambda_i, \quad \sum_{i=1}^{9} \gamma_i = 0 \tag{19}$$

in mean longitudes of all the bodies. The coefficients depend on the masses, mean motions and the semi-major axes. The intermediary was obtained up to 14th order relative to small parameters M_i, $\frac{n_i}{n}$, $\frac{\mathrm{a}}{\mathrm{a}_i}$ in (Ivanova, 2011). Its expansion contains about ten thousands terms.

4. DETERMINATION OF δp AND w

The functions $\delta p, w$ satisfy the equations

$$\delta\ddot{p} + 2\sqrt{-1}n\delta\dot{p} + n^2\left[\left(-\frac{3}{2}+K\right)\delta p + \left(-\frac{3}{2}+L\right)\delta q\right] = n^2 P' \,, \tag{20}$$

$$\ddot{w} + n^2(1+M)w = n^2 W' \tag{21}$$

with the right–hand members of the form

$$P' = P - P^{(0)} + K\delta p + L\delta q, \qquad W' = W + Mw \,. \tag{22}$$

P', W' don't contain the linear terms with respect to the lunar variables δp, δq and w.
$P^{(0)}$ is the right–hand member of the equation for the intermediary,
K, L, M are the functions of the intermediate solution.
δp and w are sought by iterations in the form of power series

$$\delta p = \sum p_{klmn} \prod_{i=1}^{9} a_i^{k_i}\overline{a}_i^{l_i} b_i^{m_i}\overline{b}_i^{n_i} \,, \qquad w = \sum w_{klmn} \prod_{i=1}^{9} a_i^{k_i}\overline{a}_i^{l_i} b_i^{m_i}\overline{b}_i^{n_i} \tag{23}$$

with the initial approximation

$$\delta p = -\frac{1}{2}a_9 + \frac{3}{2}\overline{a}_9 \,, \qquad w = b_9 + \overline{b}_9 \,. \tag{24}$$

The summation is performed over all the non-negative values of 9-indices k, l, m, n. The coefficients are the functions of the intermediate solution.

The method of δp and w construction is based on the separation of the fast and slowly changing variables by means of a number of the linear and Birkhoff transformations of the variables. The series (23) are in fact not a solution of the equations of the Moon's motion but a transformation to the secular system describing the evolution of the lunar orbit.

5. SECULAR SYSTEM

The time dependence of Laplace-type variables is determined by the solution of the autonomous secular system

$$\dot{\alpha} = \sqrt{-1}\, N\left[A\alpha + \Phi(\alpha, \overline{\alpha}, \beta, \overline{\beta})\right], \qquad \dot{\beta} = \sqrt{-1}\, N\left[B\beta + \Psi(\alpha, \overline{\alpha}, \beta, \overline{\beta})\right] \tag{25}$$

in slowly changing variables

$$\alpha = (\alpha_1, \ldots, \alpha_9)\ (\alpha_i = a_i \exp -\sqrt{-1}\lambda_i)\,,$$
$$\beta = (\beta_1, \ldots, \beta_9)\ (\beta_i = b_i \exp -\sqrt{-1}\lambda_i)\,. \tag{26}$$

Here $N = diag(n_1, \ldots, n_9)$, A and B are 9×9 matrices of semi-major axes, mean motions and masses of all the bodies under consideration, 9-vectors Φ, Ψ contain only forms of odd degree in slowly changing variables α, $\overline{\alpha}$, β, $\overline{\beta}$ starting with the third degree terms.

To complete this system one should add the equations for the conjugate variables. The right–hand members of the secular system for the Moon are obtained in the purely analytical form but the trigonometric solution of the secular system has the semi–analytical form. It includes terms due to the secular evolution of the lunar perigee and node as well as of that of the major planets.

Now this work is in progress.

6. REFERENCES

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