Post-post-Newtonian light propagation without integrating geodesic equations

Pierre Teyssandier

Dpt Système de Référence Temps et Espace, CNRS-UMR 8630

Observatoire de Paris

Introduction

- Post-post-Newtonian propagation of light will be required in future tests of GR.
- Even for the Gaia project, an analytical post-post-Newtonian solution has been proposed for light propagation in a 3-parameter family of static, spherically symmetric space-times (Klioner & Zschocke 2009 2010) :
  - ▶ obtained by integrating the geodesic equations;
  - ▶ well adapted to a ray emitted in a given direction.
- We solve the same problem by a very different method we have recently proposed (Le Poncin-Lafitte *et al.* 2004, T & Le Poncin-Lafitte 2004-2008) :
  - ▶ avoids any integration of geodesic equations;
  - ▶ well adapted to a ray emitted and observed at points both at a finite distance.
- Equivalent method (Fermat's principle) recently used by N. Ashby & B. Bertotti 2010.
- See also, X. M. Deng & T.-Y. Huang, poster *Journées 2010*.

Direction of light and travel time of photons (1)

$(\mathcal{V}_4, g)$ is a stationary space-time, covered by adapted coordinates $x^0 = ct,\ \boldsymbol{x} = (x^i)$.

Let Γ_{AB} be a light ray emitted at $\boldsymbol{x}_A$ and received at $\boldsymbol{x}_B \longrightarrow \Gamma_{AB}$ is a null geodesic.

The direction of Γ_{AB} at any of its points $x(\lambda)$ is characterized by the triple

$$\widehat{\underline{\boldsymbol{l}}} = (l_i/l_0)\,, \quad l_i = g_{i\beta}\frac{dx^\beta}{d\lambda}, \quad l_0 = g_{0\beta}\frac{dx^\beta}{d\lambda}, \quad \lambda = \text{ arbitrary parameter of } \Gamma_{AB}.$$

The triples $\widehat{\underline{\boldsymbol{l}}}_A$ and $\widehat{\underline{\boldsymbol{l}}}_B$ are given by

$$\boxed{\widehat{\underline{\boldsymbol{l}}}_A \equiv \left(\frac{l_i}{l_0}\right)_A = c\frac{\partial\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)}{\partial x_A^i}, \qquad \widehat{\underline{\boldsymbol{l}}}_B \equiv \left(\frac{l_i}{l_0}\right)_B = -c\frac{\partial\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)}{\partial x_B^i}}$$

where $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$ is defined as

$$\boxed{t_B - t_A \equiv \text{Travel time of the photon between } \boldsymbol{x}_A \text{ and } \boldsymbol{x}_B = \mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)}$$

Direction of light and travel time of photons (2)

At least three methods for determining $\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B)$:

- Integration of null geodesic equations
- Determination of Synge's world function
- Integration of one of the Hamilton-Jacobi equations in a stationary field, e.g. :

$$g^{00}(\boldsymbol{x}_A) + 2cg^{0i}(\boldsymbol{x}_A)\frac{\partial\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)}{\partial x^i_A} + c^2 g^{ij}(\boldsymbol{x}_A)\frac{\partial\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)}{\partial x^i_A}\frac{\partial\mathcal{T}(\boldsymbol{x}_A,\boldsymbol{x}_B)}{\partial x^j_A} = 0$$

Iterative algorithm for integrating this equation.

Travel time of photons in spherically symmetric space-times (1)

We consider the metrics of the form $(m = GM/c^2$, $M =$ mass of the central body)

$$ds^2 = \left(1 - \frac{2m}{r} + 2\beta\frac{m^2}{r^2} + \cdots\right)\left(dx^0\right)^2 - \left(1 + 2\gamma\frac{m}{r} + \frac{3}{2}\epsilon\frac{m^2}{r^2} + \cdots\right)\delta_{ij}dx^i dx^j,$$

$\epsilon =$ post-post-Newtonian parameter. In GR : $\beta = \gamma = \epsilon = 1$.

Travel time of photons between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$ (Le Poncin-Lafitte *et al* 2004) :

$$\mathcal{T}(\boldsymbol{x}_A, \boldsymbol{x}_B) = \frac{r_{AB}}{c} + \frac{(\gamma+1)m}{c}\ln\left(\frac{r_A + r_B + r_{AB}}{r_A + r_B - r_{AB}}\right)$$
$$+m^2\frac{r_{AB}}{c}\left[\frac{\kappa\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{x}_A \times \boldsymbol{x}_B|} - \frac{(1+\gamma)^2}{r_A r_B + (\boldsymbol{x}_A.\boldsymbol{x}_B)}\right] + \cdots,$$

where

$$r_{AB} = |\boldsymbol{x}_B - \boldsymbol{x}_A|, \quad \boldsymbol{n}_A = \frac{\boldsymbol{x}_A}{r_A}, \quad \boldsymbol{n}_B = \frac{\boldsymbol{x}_B}{r_B}, \quad \kappa = \frac{8 - 4\beta + 8\gamma + 3\epsilon}{4}.$$

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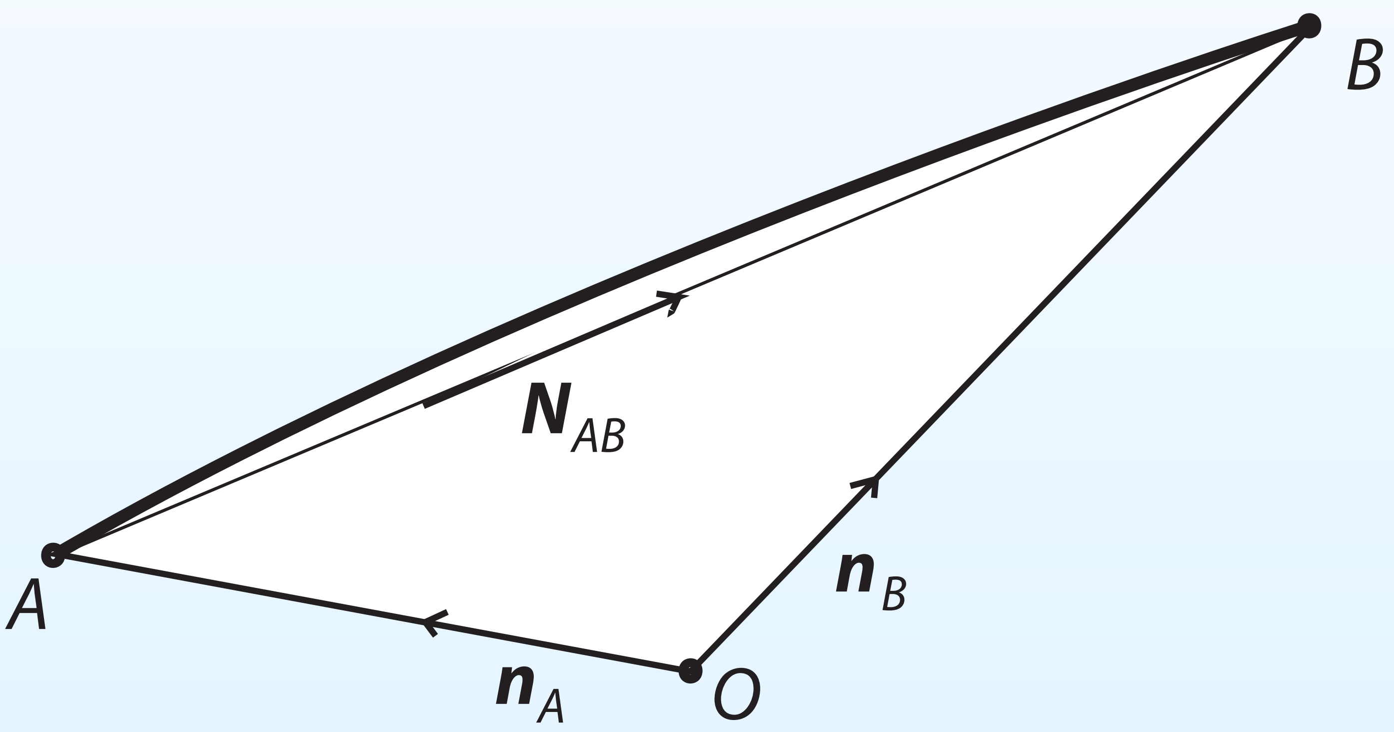
Travel time of photons in spherically symmetric space-times (2)



Propagation direction of light

Defining $\boldsymbol{N}_{AB}$ and $\boldsymbol{P}_{AB}$ as

$$\boldsymbol{N}_{AB} = \frac{\boldsymbol{x}_B - \boldsymbol{x}_A}{r_{AB}} = \frac{r_B}{r_{AB}}\boldsymbol{n}_B - \frac{r_A}{r_{AB}}\boldsymbol{n}_A, \quad \boldsymbol{P}_{AB} = \boldsymbol{N}_{AB} \times \left(\frac{\boldsymbol{n}_A \times \boldsymbol{n}_B}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\right),$$

we get

$$\begin{aligned} \widehat{\underline{\boldsymbol{l}}}_A &= -(1 + u_{AB})\boldsymbol{N}_{AB} - w_{AB}\boldsymbol{P}_{AB}, \\ \widehat{\underline{\boldsymbol{l}}}_B &= -(1 + u_{BA})\boldsymbol{N}_{AB} + w_{BA}\boldsymbol{P}_{AB}, \end{aligned}$$

where

$$u_{AB} = (\gamma + 1)\frac{m}{r_A} + \frac{m^2}{r_A^2}\left[\kappa - \frac{(\gamma+1)^2}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\right],$$

$$\begin{aligned} w_{AB} = (\gamma + 1)\frac{m}{r_A}\frac{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} + \frac{m^2}{r_A^2}\frac{1}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\Bigg\{\kappa\Bigg[\frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|}\left(1 - \frac{r_A}{r_B}\boldsymbol{n}_A.\boldsymbol{n}_B\right) \\ + \frac{r_A}{r_B} - (\boldsymbol{n}_A.\boldsymbol{n}_B)\Bigg] - (\gamma+1)^2\left(1 + \frac{r_A}{r_B}\right)\frac{1 - \boldsymbol{n}_A.\boldsymbol{n}_B}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B}\Bigg\}. \end{aligned}$$

Introduction of the impact parameter (1)

In any static, spherically symmetric, asymptotically Minkowskian space-time :

- Euler-Lagrange eqs. of geodesics $\Longrightarrow \quad \boldsymbol{L} \equiv -\boldsymbol{x} \times \underline{\widehat{\boldsymbol{l}}} = const\ vector$
- $|\boldsymbol{x}| \longrightarrow \infty \quad \Longrightarrow \quad \underline{\widehat{\boldsymbol{l}}} \longrightarrow -\left(\frac{d\boldsymbol{x}}{cdt}\right)_{\infty}$.

As a consequence

$$|\boldsymbol{L}| = lim_{|\boldsymbol{x}|\to\infty} \left| \boldsymbol{x} \times \frac{d\boldsymbol{x}}{cdt} \right| = b = \text{ impact parameter of the ray.}$$

b = Euclidean distance between the asymptote and a parallel straight line passing through the center O *as measured by an inertial observer at rest at infinity*. So

$$\boxed{b = |-\boldsymbol{x} \times \underline{\widehat{\boldsymbol{l}}}| \ \textit{ is an intrinsic quantity}}$$

See also Chandrasekhar 1983.

Introduction of the impact parameter (2)

The impact parameter b of the ray Γ_{AB} is then given by

$$b = | - \boldsymbol{x}_A \times \widehat{\underline{\boldsymbol{l}}}_A | = | - \boldsymbol{x}_B \times \widehat{\underline{\boldsymbol{l}}}_B |,$$

that is

$$\begin{aligned} b = \frac{r_A r_B}{r_{AB}} |\boldsymbol{n}_A \times \boldsymbol{n}_B| \Bigg\{ & 1 + (\gamma+1) \frac{m}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} \left(\frac{1}{r_A} + \frac{1}{r_B} \right) \\ & + \frac{m^2}{r_A r_B} \left[\kappa \frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|} - \frac{(\gamma+1)^2}{1 + \boldsymbol{n}_A.\boldsymbol{n}_B} \right] \\ & + \frac{m^2}{r_A r_B} \frac{r_{AB}}{r_A r_B} \left\{ \frac{\kappa}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|^2} \left[1 - \frac{\arccos(\boldsymbol{n}_A.\boldsymbol{n}_B)}{|\boldsymbol{n}_A \times \boldsymbol{n}_B|} (\boldsymbol{n}_A.\boldsymbol{n}_B) \right] - \frac{(\gamma+1)^2}{(1 + \boldsymbol{n}_A.\boldsymbol{n}_B)^2} \right\} \Bigg\}. \end{aligned}$$

It is the thorough expression of b within the post-post-Newtonian approximation.

Only the 1st-order approximation is useful here.

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Light direction in terms of impact parameter

Using the expression of the impact parameter b and putting

$$\phi_A = \text{angle}\,(\boldsymbol{N}_{AB}, \boldsymbol{n}_A), \quad \phi_B = \text{angle}\,(\boldsymbol{N}_{AB}, \boldsymbol{n}_B), \quad \psi_{AB} = \text{angle}\,(\boldsymbol{x}_A, \boldsymbol{x}_B),$$

we get

$$\begin{aligned} \widehat{\underline{\boldsymbol{l}}}_A &= -(1 + u_{AB})\boldsymbol{N}_{AB} - w_{AB}\boldsymbol{P}_{AB}, \\ \widehat{\underline{\boldsymbol{l}}}_B &= -(1 + u_{BA})\boldsymbol{N}_{AB} + w_{BA}\boldsymbol{P}_{AB}, \end{aligned}$$

where

$$\begin{aligned} u_{BA} &= \frac{m}{b}\left\{\gamma + 1 + \frac{m}{b}\left[\kappa \sin\phi_B + (\gamma+1)^2 \frac{\sin\phi_A}{1+\cos\psi_{AB}}\right]\right\}\sin\phi_B, \\ w_{BA} &= \frac{m}{b}\left\{(\gamma+1)\frac{\sin\psi_{AB}}{1+\cos\psi_{AB}} - \kappa\frac{m}{b}\left[\frac{\psi_{AB}}{\sin\psi_{AB}}\cos\phi_A - \cos\phi_B\right]\right\}\sin\phi_B. \end{aligned}$$

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Source located at infinity

Assume that A is at infinity. We have just to take the limits of $\underline{\widehat{\boldsymbol{l}}}_A$ and $\underline{\widehat{\boldsymbol{l}}}_B$ as

$$r_A \to \infty, \qquad \boldsymbol{n}_A \to -\boldsymbol{N}_{AB}, \qquad \boldsymbol{P}_{AB} \to -\boldsymbol{N}_{AB} \times \left(\frac{\boldsymbol{N}_{AB} \times \boldsymbol{n}_B}{|\boldsymbol{N}_{AB} \times \boldsymbol{n}_B|} \right),$$

where $\boldsymbol{N}_{AB}$ = direction of emission at infinity. Thus

$$\underline{\widehat{\boldsymbol{l}}}_A = -\boldsymbol{N}_{AB},$$

$$\underline{\widehat{\boldsymbol{l}}}_B = -\left\{ 1 + \frac{m}{b}\left[\gamma + 1 + \kappa \frac{m}{b}\sin\phi_B\right]\sin\phi_B \right\}\boldsymbol{N}_{AB} + \frac{m}{b}\left\{ (\gamma+1)(1+\cos\phi_B) + \kappa\frac{m}{b}\left[\pi - \phi_B + \frac{1}{2}\sin 2\phi_B\right]\right\}\boldsymbol{P}_{AB},$$

with

$$b = r_c\left[1 + \frac{(\gamma+1)m}{r_c}\,\frac{\sin\phi_B}{1-\cos\phi_B} + O\left(1/c^4\right)\right], \qquad r_c = r_B \sin\phi_B .$$

Deflection of light coming from infinity

Deflection of light at $\boldsymbol{x}_B$:

$$\delta\chi_B = \text{ angle } (\boldsymbol{N}_{AB}, -\underline{\widehat{\boldsymbol{l}}}_B) = \frac{|\boldsymbol{N}_{AB} \times \underline{\widehat{\boldsymbol{l}}}_B|}{|\underline{\widehat{\boldsymbol{l}}}_B|}.$$

Since $m = GM/c^2$, we find

$$\boxed{\delta\chi_B = \underbrace{\frac{(\gamma+1)GM}{c^2 b}(1+\cos\phi_B)}_{} + \frac{G^2M^2}{c^4b^2}\left[\kappa\left(\pi - \phi_B + \frac{1}{2}\sin 2\phi_B\right) - (\gamma+1)^2 \sin\phi_B (1+\cos\phi_B)\right]}$$

with (see the previous slide)

$$\boxed{b = r_B \sin\phi_B \left[1 + \frac{(\gamma+1)GM}{c^2 r_B}\frac{1}{1-\cos\phi_B} + \cdots\right]}$$

The term $\underbrace{\ldots}$ is currently used in highly precise astrometry.

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Coordinate enhanced post-post-Newtonian term

If b is replaced by its coordinate expression, then $\delta\chi_B$ may be written as

$$\delta\chi_B = \frac{(\gamma+1)GM}{c^2 r_c}(1+\cos\phi_B) + \frac{G^2M^2}{c^4 r_c^2}\underbrace{\left\{\kappa\left(\pi-\phi_B+\frac{1}{2}\sin 2\phi_B\right) -2(\gamma+1)^2\frac{\cos^3(\phi_B/2)}{\sin(\phi_B/2)}[1+2\sin^2(\phi/2)]\right\}}.$$

For a ray grazing a planet of radius $r_e \ll r_B$, $\{\underbrace{\ldots}\}$ → post-post-Newtonian contribution

$$\delta\chi^{(2)}_{graze} \approx -4(\gamma+1)^2(m/r_e)^2(r_B/r_e).$$

For Jupiter, $\delta\chi^{(2)}_{graze} \approx 16.1\,\mu$as with $r_B = 6$ AU : it is “great" for Gaia! However, this term is a pure coordinate effect... → confirmation of the analysis of Klioner & Zschocke 2010.

Total deflection of light

Owing to the symetry of the field, the total deflection $\Delta\chi$ of light is given by

$$\Delta\chi = 2\delta\chi_B$$

when B is the pericenter of the ray.

If B is the pericenter, then

$$\underline{\widehat{\boldsymbol{l}}}_B.\boldsymbol{n}_B = 0 \qquad \Longleftrightarrow \qquad \phi_B \equiv \text{ angle}(\boldsymbol{N}_{AB}, \boldsymbol{n}_B) = \frac{\pi}{2} - \frac{\Delta\chi}{2}.$$

The equation obtained for $\delta\chi_B$ yields by an iterative process

$$\boxed{\Delta\chi = \frac{2(\gamma+1)GM}{c^2 b} + \frac{(8-4\beta+8\gamma+3\epsilon)\pi G^2 M^2}{4c^4 b^2} + \cdots}$$

Given by Klioner and Zschocke 2010. See also Xie & Huang 2008 (but not intrinsic).

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Concluding remarks

- We give a new analytical determination of the propagation direction of light in static, spherically symmetric space-times within the post-post-Newtonian approximation.

- We show that the methods derived from the variational definition of null geodesics lead to straightforward calculations.

- Our results constitute an entirely independent confirmation of a recent discussion carried out by Klioner & Zschocke in the context of the Gaia project.

- It may be emphasized that the triples $\underline{\widehat{\boldsymbol{l}}}_A$ and $\underline{\widehat{\boldsymbol{l}}}_B$ are also sufficient to carry out the calculation of the frequency transfers between $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$.

References

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