About the MacCullagh relations in Relativity

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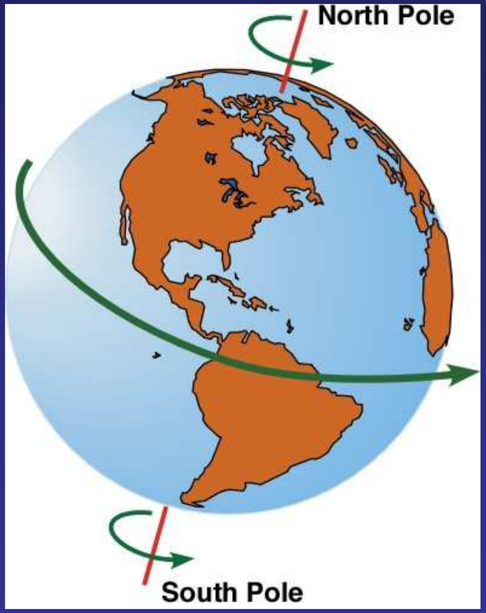
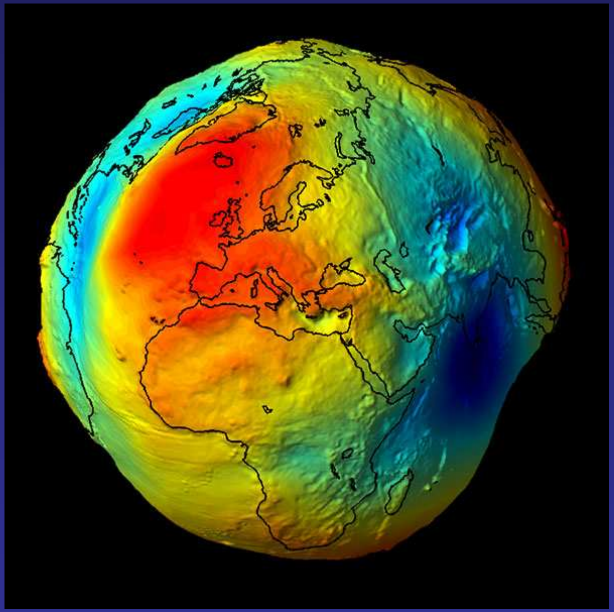
MacCullagh relations in the Newtonian framework



Earth‘s rotation



Earth‘s gravity field

Newtonian MacCullagh relations: spherical representation

$$
\begin{aligned}
C_{21} &= -\frac{I_{13}}{Ma^2} \\
S_{21} &= -\frac{I_{23}}{Ma^2} \\
C_{22} &= \frac{I_{22} - I_{11}}{4Ma^2} \\
S_{22} &= -\frac{I_{12}}{2Ma^2} \\
C_{20} &= \frac{1}{Ma^2}\left(\frac{I_{11} + I_{22}}{2} - I_{33}\right)
\end{aligned}
$$

$$U(\mathbf{x}) = \left(\frac{GM}{r}\right) \sum_{l=0}^{\infty} \sum_{m=0}^{l} \left(\frac{a}{r}\right)^{l} P_{lm}(\cos\theta)(C_{lm}\cos m\phi + S_{lm}\sin m\phi)$$

$$I_{ij} = \int_B d^3x\,\rho(\mathbf{x}^2\delta_{ij} - x^i x^j)$$

Newtonian MacCullagh relations: Cartesian coordinates

$$M_{ij} = -\mathrm{STF}(I_{ij})$$

$$U(\mathbf{x}) = G \sum_{l=0}^{\infty} \frac{(2l-1)!!}{l!} M_L \frac{\hat{n}_L}{r^{l+1}}$$

$$\hat{n}_L \equiv \frac{\hat{x}^L}{r^l}$$

$$\hat{x}^L \equiv \hat{x}^{i_1 \dots i_l} = \mathrm{STF}_{i_1 \dots i_l}\left(x^{i_1} \cdots x^{i_l}\right)$$

$$\hat{x}^{ij} \equiv \mathrm{STF}_{i,j}(x^i x^j) = x^i x^j - \frac{1}{3}\delta_{ij}\mathbf{x}^2$$

Relativity

Post-Newtonian framework (DSX)

Blanchet-Damour quadrupole mass moment

$$M_{ij} = \int_B d^3x \hat{x}_{ij}\sigma + \frac{1}{14c^2}\frac{d^2}{dt^2}\int_B d^3x \hat{x}_{ij}\mathbf{x}^2\sigma - \frac{20}{21c^2}\frac{d}{dt}\int_B d^3x \hat{x}^{ijk}\sigma^k$$

$$\sigma \equiv \frac{T^{00} + T^{ss}}{c^2}\,, \quad \sigma^i \equiv \frac{T^{0i}}{c}$$

A post-Newtonian spin vector was defined in DSX III

$$S_i = \epsilon_{ijk} \int_B d^3x \, x^j \left[\sigma^k \left(1 + \frac{4U}{c^2} \right) - \frac{\sigma}{c^2} \left(4V^k - \frac{1}{2} Q^k \right) \right]$$

$$\begin{aligned} V^k(\mathbf{x}) &\equiv G \int_B d^3x' \, \frac{\sigma^k(\mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} \\ Q^k(\mathbf{x}) &\equiv G \int_B d^3x' \, \sigma^l(\mathbf{x}') \frac{\delta_{kl} - n^k_{xx'} n^l_{xx'}}{|\mathbf{x} - \mathbf{x}'|} \end{aligned}$$

$$n^k_{xx'} \equiv \frac{(x - x')^k}{|\mathbf{x} - \mathbf{x}'|}$$

From this a post-Newtonian moment of inertia tensor can be derived

Question

Is the Newtonian relation

$$M_{ij} = -\mathrm{STF}(I_{ij})$$

valid also to post-Newtonian order ?

To answer this questions we employed a specific Newtonian model for the 1/c2 terms:

a homogeneous oblate spheroid with b (in z-direction) <=

the internal graviational potential U is given by

$$U(\mathbf{x}) = \pi G\rho \left[\mathcal{I} - \mathcal{A}(x^2 + y^2) - \mathcal{B}z^2\right]$$

where

$$\begin{aligned}
\mathcal{A} &= \frac{(1-e^2)^{1/2}}{e^3}\arcsin(e) - \frac{1-e^2}{e^2} = \frac{2}{3} - \frac{2}{15}e^2 - \frac{8}{105}e^4 + \ldots \\
\mathcal{B} &= \frac{2}{e^2} - \frac{2(1-e^2)^{1/2}}{e^3}\arcsin(e) = \frac{2}{3} + \frac{4}{15}e^2 + \frac{16}{105}e^4 + \ldots
\end{aligned}$$

$(2\mathcal{A} + \mathcal{B} = 2)$ and

$$\mathcal{I} = 2\mathcal{A}a^2 + \mathcal{B}b^2\,.$$

The geometrical eccentricity e of the spheroid is given by

$$e^2 = 1 - \frac{b^2}{a^2}\,.$$

RESULT

$$M_{ij} \neq -\mathrm{STF}(I_{ij})$$

For our model:

$$M_{ij} = -\mathrm{STF}(I_{ij}) - \left(\frac{148}{875}e^2\right)\left(\frac{GM}{c^2 a}\right)(Ma^2)\,\hat{e}_{ij} + \mathcal{O}(e^4)$$

$$\hat{e}_{ij} \equiv \mathrm{diag}(1, 1, -2)$$

PN MacCullagh relations, spherical representation

For our specific model only one MC relation needs to be modified

$$C_{20}^{*} = \frac{1}{Ma^2}\left(\frac{I_{11}+I_{22}}{2} - I_{33}\right)$$

$$C_{20}^{*} = C_{20} - \frac{444}{875}\left(\frac{GM_{\odot}}{c^2 a}\right) e^2$$

δ - term

Numbers for the Earth:

$$e^2 = 0.0066944$$

Taking the EGM96 value for $GM_\oplus$ $(3.986004418 \cdot 10^{14}\,\mathrm{m}^3\mathrm{s}^{-2})$ the relativistic scale factor reads

$$\left(\frac{GM_\oplus}{c^2 a}\right) = 6.9534851 \cdot 10^{-10}$$

so that

$$\delta \equiv \frac{444}{875}\left(\frac{GM_\oplus}{c^2 a}\right) e^2 = 2.362 \times 10^{-12}$$

Implications for a post-Newtionian SMART-97 like nutation model

In SMART97 GEM-T3 values for the potential coefficients are used:

$$\begin{aligned} C_{20} &= -1082.626\,074\,59 \times 10^{-6} \\ C_{22} &= 1.574\,410\,20 \times 10^{-6} \\ S_{22} &= -\,0.903\,757\,18 \times 10^{-6} \,. \end{aligned}$$

The principle moments of interia, A, B and C are obtained in the following way: one starts with a certain initial value for C:

$$C = 0.180\,548\,385\,370 \times 10^{-14}\ M_\odot \mathrm{AU}^2 \,,$$

where $M_\odot$ is the mass of the Sun,

$$\frac{M_\odot}{M_\oplus} = 332\,946.045 \,,$$

and the astronomical unit, AU, was taken as

$$\mathrm{AU} = 149\,597\,870\,610\,\mathrm{m} \,.$$

$B - A$ is then obtained from

$$\frac{B-A}{M_\oplus a^2} = 4\sqrt{C_{22}^2 + S_{22}} = 7.261\,454 \times 10^{-6}$$

that would still be valid relativistically (note, that that the potential coefficients refer to axes that differ in the equatorial plane by an angle $\alpha = 14^\circ.95$ (West) with respect to the principle axes of inertial). Then the Newtonian MacCullagh relation for the dynamical ellipticity H_d is used for SMART97:

$$H_d \equiv \frac{2C - A - B}{2C} = -M_\oplus a^2 \frac{C_{2,0}}{C}$$

that leads to

$$\begin{aligned} A &= 0.179955329763 \times 10^{-14}\ M_\odot \mathrm{AU}^2 \\ B &= 0.179959294245 \times 10^{-14}\ M_\odot \mathrm{AU}^2 . \end{aligned}$$

If the post-Newtonian MacCullagh relation (30) is used one would get

$$\begin{aligned} A &= 0.179955329761 \times 10^{-14}\ M_\odot \mathrm{AU}^2 \\ B &= 0.179959294243 \times 10^{-14}\ M_\odot \mathrm{AU}^2 . \end{aligned}$$

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12th digit
after the comma

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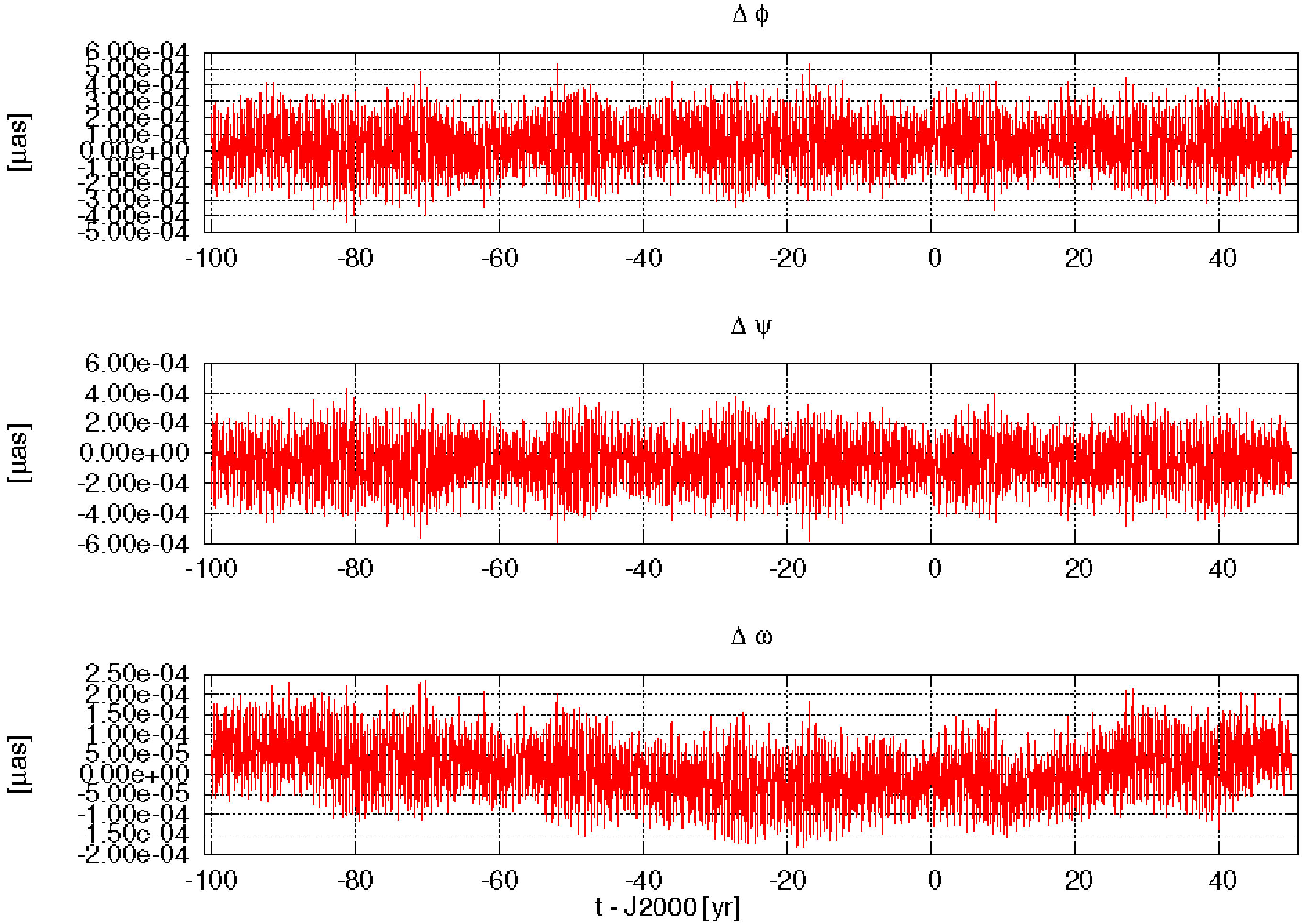

Figure 1: Differences (without δ/ with δ) for the three Euler angles $\Delta\phi$, $\Delta\psi$ and $\Delta\omega$ are shown as functions of time. The δ was magnified by a factor of 1000, so that effects for the real Earth will be about one thousand times smaller.

CONCLUSION

A violation of the classical Mac Cullagh relations in Relativity can safely be neglected for a post-Newtonian Prec/Nut theory at 0.1 μas accuracies