ON SOLUTION OF THE THREE-AXIAL EARTH'S ROTATION PROBLEM

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The rigid Earth's rotation problem of the three axial Earth is treated in the form compatible with the General Planetary Theory GPT (Brumberg, 1995).

EQUATIONS OF MOTION

The kinematical and dynamical equations of the Earth's rotation in terms of Euler angles ψ, θ, φ

$$\omega_1 + \mathrm{i}\,\omega_2 = (-\dot\theta + \mathrm{i}\,\dot\psi\sin\theta)\exp(-\mathrm{i}\,\varphi), \quad \omega_3 = \dot\varphi - \dot\psi\cos\theta \quad (1)$$

and

$$I_i\dot\omega_i - \sum_{j,k=1}^{3} \varepsilon_{ijk} I_j\omega_j\omega_k = M_i \qquad (i = 1,2,3) \qquad (2)$$

with principal inertia moments I_i ($I_1 \leq I_2 b I_3$ for the Earth) and

vector $\boldsymbol{\omega} = (\omega_i)$ of the Earth's rotation angular velocity with components ω_i referred to a rotating, Earth–fixed system $\mathbf{x}' = (x_i')$ oriented by means of the principal axes of inertia.

ε_{ijk} stands for the Levi-Civita symbol

$$\varepsilon_{ijk} = \frac{1}{2}(i-j)(j-k)(k-i).$$

The torque vector $\mathbf{M} = (M_i)$ is determined by

$$M_1 + \mathrm{i}\, M_2 = \left[\frac{\mathrm{i}}{\sin\theta} \left(\frac{\partial U}{\partial \psi} + \cos\theta \frac{\partial U}{\partial \varphi} \right) - \frac{\partial U}{\partial \theta} \right] \exp(-\mathrm{i}\,\varphi),$$

$$M_3 = \frac{\partial U}{\partial \varphi} \tag{3}$$

with the force function U.

Euler parameters.

$$\mathbf{u} = (u_1, u_2, u_3, u_4),$$

$$u_1 = -\sin\frac{\theta}{2}\cos\frac{\psi+\varphi}{2}, \qquad u_3 = -\cos\frac{\theta}{2}\sin\frac{\psi-\varphi}{2},$$

$$u_2 = \sin\frac{\theta}{2}\sin\frac{\psi+\varphi}{2}, \qquad u_4 = \cos\frac{\theta}{2}\cos\frac{\psi-\varphi}{2}. \tag{4}$$

Complex Euler parameters.

$$u = u_1 + \mathrm{i}\, u_2\,, \;\; v = u_3 + \mathrm{i}\, u_4$$

$$u = -\sin\frac{\theta}{2}\exp\left(-\,\mathrm{i}\,\frac{\psi+\varphi}{2}\right), \quad v = \mathrm{i}\cos\frac{\theta}{2}\exp\left(\mathrm{i}\,\frac{\psi-\varphi}{2}\right). \tag{5}$$

NOTATIONS

$$\omega_1 = \Omega m_1\,, \quad \omega_2 = \Omega m_2\,, \quad \omega_3 = \Omega(1 + m_3)\,, \tag{6}$$

Ω is the mean Earth's rotation velocity.
$\Omega = 7.292115 \cdot 10^{-5}$ rad/s.
The dimensionless quantities m_i are small
$m_1 \sim m_2 \sim m_3 \sim 10^{-8}$.

$$m_1 = 2\sqrt{k_2}\, m_1'\,, \qquad m_2 = 2\sqrt{k_1}\, m_2'\,.$$

$$k_1 = \frac{I_3 - I_1}{2I_2}\,, \qquad k_2 = \frac{I_3 - I_2}{2I_1}\,.$$

$$\Omega = -2n\,, \qquad n = \text{const.} \tag{7}$$

EQUATIONS OF THE EARTH' ROTATION IN THE GPT FORM

$$\dot{X} = \mathrm{i}\,\mathcal{N}[PX + R(X,t)] \tag{8}$$

where X and R stand for 7–vectors of the variables and right–hand members, respectively,

$$X = (X_i) = (u,\ \bar{u},\ v,\ \bar{v},\ m',\ \bar{m}',\ m_3), \qquad R = (R_i), \quad (i = 1, 2, \ldots, 7) \tag{9}$$

$$R_1 = m_3 u - m\bar{v}\,, \qquad R_2 = -\bar{R}_1\,, \qquad R_3 = m_3 v + m\bar{u}\,, \qquad R_4 = -\bar{R}_3\,,$$

$$R_5 = -4\sqrt{k_1 k_2}\, m_3 m' - \frac{1}{4n^2}\left(\frac{1}{\sqrt{k_1}}\frac{M_2}{I_2} - \frac{\mathrm{i}}{\sqrt{k_2}}\frac{M_1}{I_1}\right), \qquad R_6 = -\bar{R}_5\,,$$

$$R_7 = 2\sqrt{k_1 k_2}\,\frac{I_1 - I_2}{I_3}(m'^2 - \bar{m}'^2) + \frac{\mathrm{i}}{2n^2}\frac{M_3}{I_3}\,,$$

R_7 being a pure imaginary quantity. $\mathcal{N}$ and P are $7{\times}7$ diagonal matrices of the structure

$$\mathcal{N} = \mathrm{diag}(n, n, n, n, n, n, n)$$

$$P = \mathrm{diag}(1,\ -1,\ 1,\ -1,\ -4\sqrt{k_1 k_2}\,, 4\sqrt{k_1 k_2}\,, 0).$$

THE COMPLETE SYSTEM FOR THE PLANETARY AND LUNAR MOTIONS AND EARTH'S ROTATION

$$\dot{X} = \mathrm{i}\,\mathcal{N}[PX + R(X,t)]\,, \tag{10}$$

where

$$X = (a,\, \bar{a},\, b,\, \bar{b},\, X_{37}, \ldots, X_{43}), \quad R = (R_1, \ldots, R_4,\, R_{37}, \ldots, R_{43}) \tag{11}$$

are vectors with 43 components.

$\mathcal{N}$ and P are 43×43 diagonal matrices of the structure

$$\mathcal{N} = \mathrm{diag}(N, N, N, N, n, n, n, n, n, n, n)\,,$$

$$P = \mathrm{diag}(E_{(9)}, -E_{(9)}, E_{(9)}, -E_{(9)}, 1,\, -1,\, 1,\, -1, -4\sqrt{k_1 k_2}\,, 4\sqrt{k_1 k_2}\,, 0)$$

N is 9×9 diagonal matrix of mean motions n_i,

$E_{(9)}$ is unitary matrices of dimension 9×9.

a, $\bar{a}$, b, $\bar{b}$ are eccentric and oblique variables of the Moon and planets.

SECULAR SYSTEM
FOR THE PLANETS AND THE MOON

By means of Birkhoff normalization and linear transformations the planetary and lunar system results to a secular system.

$$\dot{z}_\sigma = \mathrm{i}(\mu_\sigma z_\sigma + n_\sigma U^*_{1\sigma}),$$

$$\dot{w}_\sigma = \mathrm{i}(\nu_\sigma w_\sigma + n_\sigma U^*_{3\sigma}) \tag{12}$$

with

$$U^*_{\kappa\sigma} = (z_\sigma \delta_{\kappa 1} + w_\sigma \delta_{\kappa 3}) \sum{}^{*} \; U^{(\kappa\sigma)}_{iklm} (z_j \overline{z}_j)^{k_j} (w_j \overline{w}_j)^{m_j} \,, \tag{13}$$

$$\kappa = 1, 3;\; \sigma = 1, 2, \ldots, 9\,.$$

μ_j and ν_j $(j = 1, 2, \ldots, 9)$ are the eigenvalues of the matrices in the linear parts of the secular planetary and lunar system .

This system admits the first integrals

$$z_j \overline{z}_j = \mathrm{const}\,, \qquad w_j \overline{w}_j = \mathrm{const} \tag{14}$$

leading to straightforward integration.

SECULAR SYSTEM FOR THE EARTH' ROTATION

$$\dot{p}_1 = \mathrm{i}\, n(p_1 + F_{37}), \tag{15}$$

$$\dot{p}_3 = \mathrm{i}\, n(p_3 + F_{39}), \tag{16}$$

$$\dot{p}_5 = \mathrm{i}\, n(-4\sqrt{k_1 k_2}\, p_5 + F_{41}), \tag{17}$$

$$\dot{p}_7 = \mathrm{i}\, n\, F_{43} \tag{18}$$

$$F_{36+\kappa} = U^{*(1)}_{4+\kappa} + U^{*(2)}_{4+\kappa}\,, \qquad \kappa = 1, \ldots, 7\,. \tag{19}$$

THE RIGHT-HAND MEMBERS OF THE SECULAR SYSTEM FOR THE EARTH' ROTATION

$$U^{*(1)}_{4+\kappa} = p_\kappa(1-\delta_{\kappa 7}) \sum{}^{*} \; U^{(4+\kappa,s)}_{iklm} (p_1p_2)^{s_2}(p_3p_4)^{s_4}(p_5p_6)^{s_6} p_7^{s_7} \times$$

$$\times (w_N\overline{w}_N)^{s_4} \prod_{j=1}^{9} (z_j\overline{z}_j)^{k_j}(w_j\overline{w}_j)^{m_j} \qquad (20)$$

$$U^{*(2)}_{4+\kappa} = (1-\delta_{\kappa 7}) p_5^{\delta_{\kappa 5}} p_6^{\delta_{\kappa 6}} \sum{}^{*} \; U^{(4+\kappa,s)}_{iklm} \Big(\prod_{j=1}^{4} p_j^{s_j}\Big)(p_5p_6)^{\min\{s_5,s_6\}} p_7^{s_7} \times$$

$$\times (w_N\bar{w}_N)^{\max\{s_3-\delta_{\kappa 3},\, s_4-\delta_{\kappa 4}\}} \prod_{j=1}^{9} (z_j\overline{z}_j)^{k_j}(w_j\overline{w}_j)^{m_j}. \qquad (21)$$

THE FIRST INTEGRALS OF THE SECULAR SYSTEM FOR THE EARTH' ROTATION

$$p_1 p_2 + (w_N \bar{w}_N) p_3 p_4 = C_1 \,, \tag{22}$$

$$p_5 p_6 = C_2 \,, \tag{23}$$

$$p_7 = C_3 \tag{24}$$

with real constants $C_1 \,, C_2 \,, C_3$.

PRACTICAL FORM OF THE SECULAR SYSTEM FOR THE EARTH' ROTATION

$$\dot{g} = \mathrm{i}\, n[gG(g\overline{g},\, h\overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j) + \Phi(g,\, \overline{g},\, h,\, \overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j)], \quad (25)$$

$$\dot{h} = \mathrm{i}\, n[hH(g\overline{g},\, h\overline{h},\, z_j\overline{z}_j,\, wj\overline{w}_j) + \Psi(g,\, \overline{g},\, h,\, \overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j)], \quad (26)$$

$$\dot{f} = \mathrm{i}\, n[fF(g\overline{g},\, h\overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j) + \Theta(g,\, \overline{g},\, h,\, \overline{h},\, z_j\overline{z}_j,\, w_j\overline{w}_j)], \quad (27)$$

where

$$p_1 = g,\;\; p_3 = h,\;\; p_5 = f,$$

$$gG = g + U_5^{*(1)}, \quad hH = h + U_7^{*(1)}, \quad fF = -4\sqrt{k_1 k_2}\, f + U_9^{*(1)}, \quad (28)$$

$$\Phi = U_5^{*(2)}, \qquad \Psi = U_7^{*(2)}, \qquad \Theta = U_9^{*(2)}. \quad (29)$$

SOLUTION OF THE EARTH' ROTATION SECULAR SYSTEM

$$\Phi = \Psi = \Theta = 0$$

Trigonometrical solution:

$$g = g_0 \exp \mathrm{i}\,\xi\,, \quad h = h_0 \exp \mathrm{i}\,\eta\,, \quad f = f_0 \exp \mathrm{i}\,\zeta\,,$$

$$\xi = n\Delta t + \xi_0\,, \quad \eta = n\sigma t + \eta_0\,, \quad \zeta = n\chi t + \zeta_0 \qquad (30)$$

with real constants g_0, h_0, , f_0, ξ_0, η_0, ζ_0 and the frequency factors

$$\Delta = G\,, \qquad \sigma = H\,, \qquad \chi = F\,. \qquad (31)$$

By combining three frequencies n, $n\Delta$, $n\sigma$ one can restore the fundamental frequencies of the classical solution.

$$n\Delta + n\sigma = \dot{\phi}\,, \qquad n\Delta - n\sigma = \dot{\psi}\,, \qquad (32)$$

The frequency χ corresponds to the Euler period of the Earth' rotation.

To evaluate the influence of Φ, Ψ, Θ one may retain the form (30) with constant Δ, σ, χ and slowly change g_0, h_0, χ, ξ_0, η_0, ζ_0.

SOLUTION OF THE EARTH' ROTATION SECULAR SYSTEM
The influence of the Φ, Ψ, Θ terms

$$\mathrm{i}\, g\dot{\xi}_0 + g g_0^{-1}\dot{g}_0 = \mathrm{i}\, n[\Phi + g(G-\Delta)], \tag{33}$$

$$\mathrm{i}\, h\dot{\eta}_0 + h h_0^{-1}\dot{h}_0 = \mathrm{i}\, n[\Psi + h(H-\sigma)]. \tag{34}$$

$$\mathrm{i}\, f\dot{\zeta}_0 + f f_0^{-1}\dot{f}_0 = \mathrm{i}\, n[\Theta + f(F-\chi)]. \tag{35}$$

$$\dot{g}_0 = n h_0 w_N \overline{w}_N \left[D\Sigma' \sin(\eta-\xi) - g_0 h_0 w_N \overline{w}_N C_1 \Sigma'' \sin(2(\eta-\xi))\right], \tag{36}$$

$$\dot{h}_0 = -n g_0 \left[D\Sigma' \sin(\eta-\xi) - g_0 h_0 w_N \overline{w}_N C_1 \Sigma'' \sin(2(\eta-\xi))\right], \tag{37}$$

$$\dot{f}_0 = 0. \tag{38}$$

$$D = g^2\overline{g}^2 - h^2\overline{h}^2 (w_N \overline{w}_N)^2,$$

Σ', Σ'' are the function of the planetary and lunar coordinates.

NUMERICAL ESTIMATES

$$n = -3.1501\,9368/d. \tag{39}$$

First integrals of Earth' secular system:

$$C_1 = 1.0000000000000,$$
$$C_2 = .0000000000003,$$
$$C_3 = .0000000201237. \tag{40}$$

NUMERICAL ESTIMATES

The frequencies:

$$\Delta = 1.0000002243152\,, \quad \sigma = 1.0000000113376\,,$$

$$\chi = -.0065689468461. \tag{41}$$

$$n(\Delta+\sigma) = -6.3003881023519/d\,, \qquad n(\Delta-\sigma) = -.0000006709206/d\,,$$

$$n\chi = .0206934548388. \tag{42}$$

Our results in $\dot\phi$ and $\dot\psi$ coincide with the SMART97 solution up to $10^{-7}/d$.

The values of $g_0^{(0)}$, $h_0^{(0)}$, $f_0^{(0)}$, $\xi_0^{(0)}$, $\eta_0^{(0)}$, $\zeta_0^{(0)}$ (for the epoch J2000):

$$g_0^{(0)} = -0.2031\,0924\,, \quad h_0^{(0)} = 70.7495\,9950\,,$$

$$f_0^{(0)} = .0000005780662 \tag{43}$$

$$\xi_0^{(0)} = -2.4474155174197\,, \quad \eta_0^{(0)} = 2.5467010955442\,,$$

$$\zeta_0^{(0)} = -.0313831496370\,. \tag{44}$$

Six constants $g_0^{(0)}$, $h_0^{(0)}$, $f_0^{(0)}$, $\xi_0^{(0)}$, $\eta_0^{(0)}$, $\zeta_0^{(0)}$ are arbitrary constants of the trigonometrical solution of the Earth' rotation secular system.

CONCLUSION

The technique of this paper allows to construct a general theory of motion and rotation of the solar system bodies.

BIRKHOFF NORMALIZATION

$$X = Y + \Gamma(Y, t) \tag{12}$$

$$Y = (a, \bar{a}, b, \bar{b}, Y_{37}, \ldots, Y_{43})$$

$$\dot{Y} = \mathrm{i}\,\mathcal{N}[PY + F(Y, t)]. \tag{13}$$

$$U = R - \mathcal{N}^{-1}\Gamma_Y \mathcal{N} U^*\,, \tag{14}$$

$$\Gamma_t + \mathrm{i}(\Gamma_Y \mathcal{N} P Y - \mathcal{N} P \Gamma) = \mathrm{i}\,\mathcal{N} U^+\,, \tag{15}$$

$$U = U^* + U^+\,, \qquad F = U^*\,. \tag{16}$$